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Investigation of free vibration of multi-cracked nonlocal nanobeams

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Abstract: This paper investigates the free transverse vibration characteristics of nanobeams containing multiple cracks using the Euler–Bernoulli beam framework augmented by Eringen’s nonlocal elasticity theory. A size-dependent model is formulated by introducing discontinuities in slope and transverse displacement at cracked sections, which are represented by rotational springs. The nonlocal elasticity is incorporated via an integral constitutive relation, leading to an enriched governing differential equation. Analytical expressions for natural frequencies and mode shapes are derived by enforcing kinematic and continuity conditions across crack locations and beam boundaries. The analysis reveals that the presence of multiple cracks significantly alters dynamic behavior, notably lowering natural frequencies compared to intact and local-behavior predictions. The effect is further amplified by increasing nonlocal parameter and crack severity. The proposed method is applicable for arbitrary crack configurations and boundary conditions, providing a versatile tool for assessing vibrational responses in nanobeams. Results are validated against existing solutions and serve as benchmarks for future theoretical and computational studies.

Keywords: cracked nanobeams, nonlocal elasticity, free vibration, analytical analysis.

1. Introduction

Nanobeams are fundamental components in the design of nanoelectromechanical systems (NEMS) and other nanodevices, where accurate modeling of their dynamic behavior is essential [1-4]. At nanoscale dimensions, size-dependent phenomena become significant, and interactions between distant material points cannot be ignored. To account for this, Eringen’s nonlocal elasticity theory [5, 6] augments classical continuum mechanics by allowing the stress at a point to be influenced by the state of strain in a surrounding domain, capturing intrinsic scale effects. Arash and Wang (2012) [7] provided a comprehensive review

of nonlocal continuum modeling approaches to capture the size-dependent mechanical behavior of carbon nanotubes and graphene, while Wu and Yu (2019) [8] delivered an extensive overview of mechanical analyses - covering bending, vibration, buckling, and dynamic instability - of rectangular nanobeams and single-, double-, and multi-walled carbon nanotubes, all grounded in Eringen’s nonlocal elasticity framework. Novakovic and Caric [9] presented a novel analysis of columnar stability and cross-sectional shape optimization for an axially-loaded nonlocal beam with asymmetric boundary conditions - a rolling support at one end and an elastic rotational restraint at the other -

employing Eringen's nonlocal elasticity framework together with Pontryagin's maximum principle to derive optimal, material-efficient beam configurations. Mamu and Togun (2024) [10] explored the nonlinear vibration behavior of nonlocal Euler–Bernoulli nanobeams embedded in nonlinear (Winkler–Pasternak) elastic media under thermal loading, deriving the governing equations using Hamilton's principle and applying the multiple-scale perturbation method to reveal how temperature, foundation nonlinearities, and nonlocal interactions influence vibration characteristics across both simply supported and clamped–clamped boundary conditions. Amendola, Zampoli and Luciano (2023) [11] developed a one-dimensional thermoelastic model for nanobeams that dissipate thermal energy via radiation, coupling Eringen's nonlocal Euler–Bernoulli elasticity with an extended Green–Naghdi II theory, and reveal how radiative effects induce damping in wave propagation through detailed waveform-based analysis.

The Euler–Bernoulli beam theory (EBT), a well-established framework for modeling slender structures, describes flexural vibrations through a fourth-order differential equation. When applied to nanobeams, incorporating nonlocal elasticity modifies the relationship between bending moment and curvature, leading to altered natural frequencies and mode shapes. Prior studies demonstrated that nonlocal effects tend to soften the dynamic response, particularly evident in smaller beam lengths or higher vibration modes. Hossain & Lellep (2023) [12] employed a power-series method to investigate how thermal environments affect the natural vibration behavior of axially graded nanobeams with multiple cracks, modeled via nonlocal Euler–Bernoulli theory and rotational springs at crack locations, revealing the combined influences of material gradient, crack severity, crack position, and small-scale effects on dynamic responses. Attia, Matbully & Osman et al. (2024) [13] investigated the free vibration behavior

of a double-cracked, bi-directional functionally graded (FG) Euler–Bernoulli nanobeam - with material properties varying along both length and thickness directions - using Eringen's nonlocal elasticity theory and rotational springs at crack sections, and efficiently solve the resulting model via the differential quadrature method, conducting comprehensive parametric studies on gradient indices, nonlocal effects, and crack severity and positions. Taima, El-Sayed & Friswell (2023) [14] employed Reddy's third-order shear deformation theory and Eringen's nonlocal elasticity within a finite element framework -modelling cracks via two rotational springs - and investigate how nonlocal effects, beam slenderness, crack severity and location, foundation stiffness, thermal loads, and boundary conditions influence the natural frequencies of cracked nanobeams supported by Winkler–Pasternak foundations. Eghbali, Hosseini & Pourseifi (2023) [15] conducted a comprehensive analysis of the free lateral vibrations of a cracked nano-beam using Euler–Bernoulli beam theory combined with nonlocal strain gradient theory, modeling cracks via rotational springs and investigating the effects of crack depth, location, material length scale parameters, and nonlocal interactions on natural frequencies. Scorza, Luciano, Carpinteri & Vantadori (2025) [16] proposed a novel nonlocal analytical model based on Griffith's energy criterion and linear elastic fracture mechanics to simulate the mechanical behavior of multi-cracked nanobeams under bending, efficiently capturing the stiffness degradation due to multiple cracks. Scorza et al. (2024) [17] proposed a novel nonlocal analytical model—based on the stress-driven nonlocal elasticity theory within an Euler–Bernoulli beam framework—that partitions a nanobeam into multiple segments connected by massless rotational springs at crack locations, enabling clear insight into how multiple cracks affect mechanical behavior under both Mode I and mixed-mode loading conditions. Lovisi (2023) [18] applied a

surface stress-driven nonlocal elasticity model—which integrates surface energy effects via a stress-driven formulation within the Euler–Bernoulli beam theory—to investigate how functionally graded (FG) cracked nanobeams bend, modeling cracks as rotational-spring discontinuities and deriving governing equations using the principle of virtual work to conduct parametric bending analyses. Caporale et al. (2023) [19] introduced a local–nonlocal stress-driven analytical model for multi-cracked Euler–Bernoulli nanobeams, combining local and nonlocal effects via a convex mixture parameter and enabling closed-form solutions (integral and differential) that illuminate how crack-induced rotational discontinuities and scale effects influence beam stiffness across various boundary conditions and crack configurations. Hai & Tang (2023) [20] employed a two-phase local/nonlocal stress-driven integral model - incorporating discontinuities at the crack via a convex mixture of local and nonlocal stress contributions - to analyze the size-dependent Mode I fracture behavior of a centrally cracked nanobeam, revealing how key fracture parameters like the energy release rate and stress intensity factor diminish significantly with increasing nonlocal effects. Aria, Friswell & Rabczuk (2019) [21] developed a finite element model, based on Hamilton's principle and Galerkin's weighted residual method, to investigate the thermal transverse vibration characteristics of cracked nanobeams embedded in a double-parameter nonlocal elastic foundation, where cracks are simulated by connecting two beam segments with a rotational spring at the crack location, and systematically explore the influences of crack length, crack position, thermal gradient, boundary conditions, and foundation stiffness on natural frequencies (validated against benchmark studies). Bahrami (2017) [22] introduced a wave propagation–based analytical framework grounded in nonlocal elasticity to examine free vibration behavior, as well as incident wave power

transmission and reflection, in multi-cracked Euler–Bernoulli nanobeams by deriving propagation, reflection, and transmission matrices, and analyzing how crack severity and nonlocal parameters affect both natural frequencies and energy partitioning at cracking sites. Loghmani & Yazdi (2018) [23] developed an elegant analytical wave-based framework grounded in Eringen's nonlocal elasticity, capable of capturing the free lateral vibration behavior of multi-cracked, stepped nanobeams by modeling cracks as rotational springs and beam steps as segment transitions - deriving propagation, reflection, and transmission matrices to quantify how nonlocal effects, crack characteristics, cross-section changes, and boundary masses (e.g., buckyballs) all influence natural frequencies and wave scattering responses. Hosseini-Hashemi, Fagher, Nazemnezhad & Sotoude Haghighi (2014) [24] analyzed the dynamic response of thin and thick cracked nanobeams by integrating surface energy effects (including surface elasticity, surface stress, and surface density) into a nonlocal continuum framework, demonstrating how surface phenomena and nanobeam geometry distinctly influence natural frequencies across different boundary conditions. Extensive research has been devoted to the vibration analysis of cracked micro/nanobeams, considering nonlocal and size-dependent theories, nonlinear effects, and vibration-based crack identification methods [25–29].

Beyond size effects, real nanobeams often contain multiple cracks, which further influence their vibration behavior. The common analytical approach models each crack by an equivalent rotational spring, introducing discontinuities in slope or deflection across cracked sections. This method has been extended effectively to nonlocal formulations of cracked nanobeams, enabling analysis of how crack severity, location, and count alter vibrational characteristics. Although previous studies have investigated multi-cracked

nanobeams within nonlocal elasticity frameworks, differences remain in terms of constitutive modeling approaches and solution strategies. Existing works have primarily relied on wave propagation matrices, power-series approximations, or mixed local–nonlocal stress-driven formulations. In contrast, the present study develops a fully analytical segment-wise formulation derived explicitly from Eringen's integral nonlocal elasticity model. Particular attention is devoted to the rigorous enforcement of nonlocal bending continuity across cracked interfaces, ensuring full consistency with the integral constitutive framework. This results in a unified analytical procedure applicable to arbitrary crack numbers and boundary conditions, while transparently recovering classical and intact-beam solutions as limiting cases.

Our investigation delves into the combined impacts of nonlocal elasticity and multiple cracks on nanobeam vibration. We explore how natural frequencies shift with changes in crack number, position, severity, and nonlocal parameter values. Notable phenomena such as softened frequencies, mode-dependent insensitivity to certain cracks, and crack shielding or amplification effects emerge from the analysis. The model reduces correctly to classical solutions when cracks vanish or the nonlocal parameter tends to zero, offering both theoretical consistency and practical insight. This analytical framework not only enhances understanding of dynamic behavior in defective nanobeams but also aids in the design and health monitoring of nanoscale structural elements.

2. Mathematical model

2.1. Cracked nanobeam

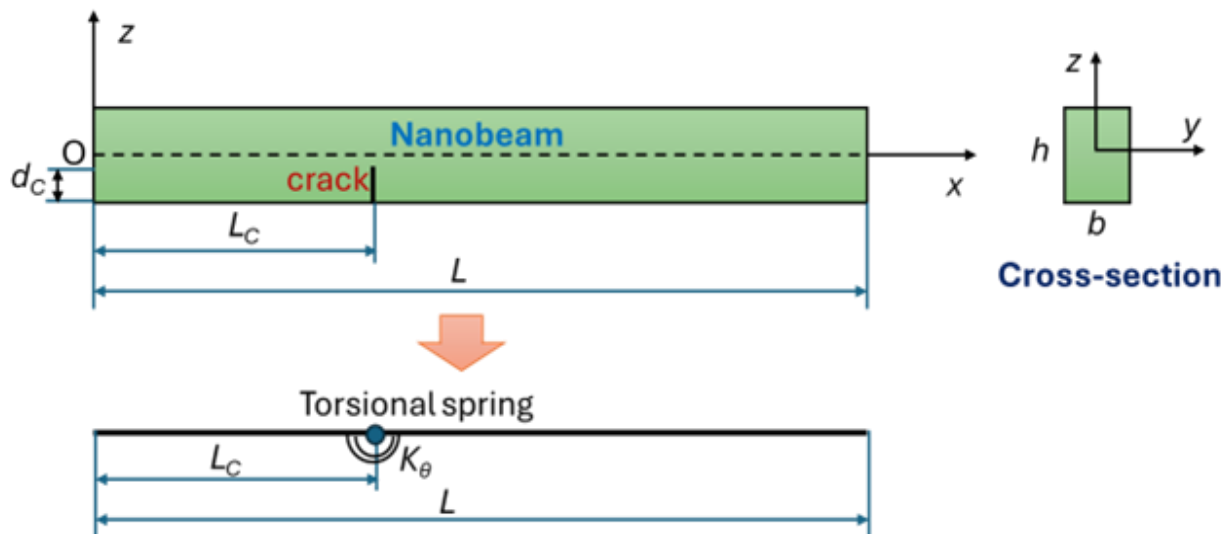


Fig. 1. Sketch of a nanobeam model containing crack

Consider a nanobeam of rectangular cross-section - with length L , thickness h , and width b - as illustrated in Fig. 1. At a distance L_c from the left end, the nanobeam contains a crack with the depth d_c . The crack is modeled by introducing a rotational spring at that section. This spring represents the local compliance induced by the crack, permitting a discontinuity in the nanobeam's rotational (slope) response under bending, while maintaining continuity of transverse displacement. The spring's stiffness K_θ is calibrated based on fracture

mechanics concepts and is directly related to the crack's severity.

When a crack is present in the nanobeam and is represented by a rotational spring, this spring stores additional elastic (strain) energy. In addition, the rotational spring itself contributes further strain energy, which is given by the classical spring energy expression:

$$U_{\text{spring}} = \frac{1}{2} K_\theta [\Delta\theta]^2, \quad (1)$$

where $\Delta\theta$ represents the discontinuity (jump) in

the rotation angle across the crack location, which are given by [25]:

$$\Delta\theta = k_{MM} \frac{\partial^2 w}{\partial x^2} + k_{MN} \frac{\partial u}{\partial x}, \quad (2)$$

where k_{MM} and k_{MN} are denoted for the flexibility constants; while $w(x,t)$ and $u(x,t)$ are transverse and axial displacements of any point on the neutral plane of the nanobeam. Given that the crossover flexibility constant k_{MN} is negligibly small compared to the bending-related flexibility constant k_{MM} [25]. We retain only k_{MM} in our formulation. Under this assumption, the rotation angle and the corresponding increase in strain energy can be expressed as follows:

$$\Delta\theta = k_{MM} \frac{\partial^2 w}{\partial x^2}, \quad (3)$$

In an Euler–Bernoulli nanobeam, the presence of an edge crack introduces a sudden change in the bending slope. At the crack location, while transverse displacement remains continuous, the rotational slope is discontinuous - reflecting the local flexibility induced by the crack. Therefore, the continuity condition at the crack section is presented as [25]:

$$\Delta\theta = \left. \frac{k_{MM}}{L} \frac{\partial^2 w}{\partial x^2} \right|_{x=L_c} = K \left. \frac{\partial^2 w}{\partial x^2} \right|_{x=L_c}, \quad (4)$$

Here, $K = k_{MM} / L$ serves as a dimensionless flexibility parameter, commonly referred to as the crack-severity constant. The torsional spring's stiffness K_θ is formulated as a decreasing nonlinear function of the crack depth ratio $\xi = d_c / h$ as below:

$$K_\theta = \frac{EI}{h} \cdot \frac{1}{\xi - \xi^2}, \quad (5)$$

In this relation, the spring stiffness approaches infinity for a vanishing crack ($d_c \rightarrow 0$) and vanishes when the crack depth equals the nanobeam's height ($d_c \rightarrow h$).

As a result, the total strain energy in the system is the sum of these two contributions:

$$U_{\text{total}} = \int_0^L \frac{M^2(x)}{2EI} dx + \frac{1}{2} K_\theta [\Delta\theta]^2. \quad (6)$$

This formulation clearly separates the energy stored due to bending of the intact nanobeam from the local flexibility introduced by the crack. It also lays a solid foundation for deriving governing equations or energy-based criteria when analyzing the vibration or stability of cracked nanobeams.

It should be emphasized that the rotational spring adopted in the present formulation represents an equivalent global flexibility model. Its purpose is to capture the effective reduction in bending stiffness induced by the crack on the overall structural response, rather than to describe the detailed crack-tip stress singularity or atomistic fracture processes. In the current nonlocal framework, the scale-dependent behavior is incorporated through Eringen's nonlocal constitutive relation governing each beam segment, while the crack flexibility parameter is calibrated from classical fracture mechanics considerations. These two effects operate at different modeling levels: nonlocal elasticity accounts for long-range interatomic interactions in the continuum description, whereas the rotational spring introduces a localized compliance representing stiffness degradation.

At the nanoscale, surface stress and surface energy effects may influence crack-tip mechanics, and enhanced models such as surface elasticity (e.g., Gurtin–Murdoch theory) could provide a more refined physical description. However, incorporating such effects would significantly increase the model complexity and shift the focus toward fracture mechanics rather than global vibration characteristics. Since the primary objective of this study is to investigate the global free vibration behavior of multi-cracked nanobeams, the equivalent rotational spring model offers a physically interpretable and analytically

tractable approach.

2.2. The nonlocal elasticity theory

Traditional (local) elasticity assumes that the stress at a point in a material depends solely on the strain at that same point. In contrast, Eringen's nonlocal elasticity theory [5, 6] embodies a paradigm shift - stress at any location is influenced by strains over an entire surrounding region. This shift acknowledges the significance of long-range. Eringen formulated a constitutive relation where stress is expressed as a weighted integral (convolution) of strain across the material domain strain-driven integral approach. Eringen's nonlocal elasticity theory has been widely utilized in nanotechnology and structural mechanics [7, 8]. It effectively models phenomena in carbon nanotubes, nanobeams, graphene sheets, and other microstructural systems. The integral form - despite paradoxes in its differential approximation - has proven valuable in many mechanical engineering applications. According to the nonlocal elasticity theory [5, 6], the normal and shear stresses of the nanobeam can be expressed as [5, 6]:

$$\sigma_{xx} = \int_V E(z)\alpha_0(x', x, e_0 a) \varepsilon'_{xx}(x') dx' \quad (7)$$

$$\sigma_{xz} = \int_V G(z)\alpha_0(x', x, e_0 a) \gamma'_{xz}(x') dx' \quad (8)$$

where, $e_0 a$ refers to as the nonlocal parameter, encapsulates the intrinsic length scale effects in Eringen's nonlocal elasticity theory. It appears in both the nonlocal kernel function $\alpha_0(x', x, e_0 a)$ and its equivalent differential operator. Here, a is the characteristic internal length - for example, interatomic spacing or particle size, while e_0 is a dimensionless material constant, which needs to be determined experimentally. The nonlocal parameter $e_0 a$ bridges micro-scale internal dimensions and macro-scale external dimensions. It quantifies how significant size-dependent (nonlocal) effects are, by comparing them to the scale of the structure.

In the differential (simplified) form, Eringen's constitutive relation is typically written as [5, 6]:

$$\left[1 - (e_0 a)^2 \nabla^2\right] \sigma_{xx} = E(z) \varepsilon_{xx} \quad (9)$$

where, ε_{xx} is the axial strain. The differential form (9) simplifies implementation (e.g., in beam or plate theory), but its equivalence to the integral form demands careful handling of boundary conditions, especially for finite or complex geometries. This constitutive framework is foundational in modeling nanobeams, nanoplates, dislocations, and other phenomena where classical elasticity fails to account for microstructural influences.

2.3. Applying the nonlocal elasticity theory to cracked nanobeam

In this work, the Euler-Bernoulli beam theory is utilized to model the cracked nanobeam. This theory is based on three key simplifying assumptions: (1) Straightness: Cross-sections of the nanobeam remain straight (no warping); (2) Inextensibility: Those cross-sections remain inextensible (i.e., do not stretch); (3) Normality: Cross-sections remain perpendicular to the neutral axis even after bending. These assumptions significantly simplify the behavior of the beam under bending. Based on this theory, the displacements of the nanobeam are expressed in the following form:

$$\begin{aligned} u_1(x, z, t) &= u(x, t) - z \frac{\partial w(x, t)}{\partial x}, \\ u_2(x, z, t) &= 0, \\ u_3(x, z, t) &= w(x, t). \end{aligned} \quad (10)$$

in which, u_1 , u_2 , and u_3 are denoted for the displacement components in x , y , and z direction, respectively. In the context of nanoscale structures, it is worth noting that transverse shear deformation and rotary inertia effects—accounted for in Timoshenko beam theory—may become significant for moderately thick beams or for structures with small aspect ratios. However, the present study focuses on slender nanobeams with relatively large length-to-thickness ratios (L/h), for

which bending deformation dominates the dynamic response and shear effects remain negligible. It should also be emphasized that the scale-dependent behavior considered here arises from Eringen's nonlocal elasticity theory, which modifies the constitutive relation to capture long-range interatomic interactions. These nonlocal effects are fundamentally different from shear deformation effects, which are primarily governed by geometric slenderness. Therefore, for slender nanobeams undergoing flexural vibration, the use of the nonlocal Euler–Bernoulli beam framework remains physically consistent and computationally efficient, while retaining the essential size-dependent characteristics. The nonzero axial strain can be determined as:

$$\varepsilon_{xx} = \frac{\partial u(x,t)}{\partial x} - z \frac{\partial w(x,t)}{\partial x}. \quad (11)$$

To derive the governing equations of motion for the cracked nanobeam, the Hamilton's principle is employed:

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W) dt = 0. \quad (12)$$

where, δT , δU and δW are the virtual kinetic energy, the virtual strain energy and the virtual work done by the external loads, respectively. The virtual strain energy of the nanobeam can be expressed as:

$$\delta U = \int_V \sigma_{xx} \delta \varepsilon_{xx} dV = \int_0^L \left(N_{xx} \frac{\partial \delta u}{\partial x} - M_{xx} \frac{\partial^2 \delta w}{\partial x^2} \right) dx. \quad (13)$$

Here, N_{xx} and M_{xx} are the stress resultants related to the axial force and bending moment, respectively, which are defined as:

$$N_{xx} = \int_A \sigma_{xx} dA, \quad M_{xx} = \int_A \sigma_{xx} z dA. \quad (14)$$

The virtual kinetic energy of the nanobeam can be expressed as:

$$\begin{aligned} \delta T = & \int_0^L \rho(z) \frac{\partial u_1}{\partial t} \delta \left(\frac{\partial u_1}{\partial t} \right) dA dx \\ & + \int_0^L \rho(z) \frac{\partial u_3}{\partial t} \delta \left(\frac{\partial u_3}{\partial t} \right) dA dx \end{aligned} \quad (15)$$

$$= \int_0^L I_0 \frac{\partial u}{\partial t} \delta \left(\frac{\partial u}{\partial t} \right) dx + \int_0^L I_0 \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) dx.$$

where $\rho(z)$ is the mass density of the nanobeam and I_0 is the mass inertia of the nanobeam which is defined as:

$$I_0 = b \int_{-h/2}^{h/2} \rho(z) dz. \quad (16)$$

The virtual work done by the external load acting on the nanobeam can be determined as:

$$\delta W = - \int_0^L q(x,t) \delta w dx. \quad (17)$$

Herein, $q(x,t)$ is the transverse distributed load acting on the nanobeam applied per unit length and induces bending in the nanobeam's deflection field $w(x,t)$.

Now, substituting Eqs. (13), (15) and (17) into Eq. (12), by performing integration by parts, the equation of motion of the nanobeam can be obtained as follows:

$$\frac{\partial N_{xx}}{\partial x} = I_0 \frac{\partial^2 u}{\partial t^2}, \quad (18)$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + q = I_0 \frac{\partial^2 w}{\partial t^2}. \quad (19)$$

The axial force and moment resultants, N_{xx} and M_{xx} , can be obtained as below:

$$N_{xx} - (e_0 a)^2 \frac{\partial^2 N_{xx}}{\partial x^2} = A_{11} \frac{\partial u}{\partial x}, \quad (20)$$

$$M_{xx} - (e_0 a)^2 \frac{\partial^2 M_{xx}}{\partial x^2} = D_{11} \frac{\partial^2 w}{\partial x^2}. \quad (21)$$

where

$$A_{11} = b \int_{-h/2}^{h/2} E(z) dz; \quad D_{11} = b \int_{-h/2}^{h/2} E(z) z^2 dz. \quad (22)$$

From Eqs. (18)-(21), the axial force and bending moment resultants can be obtained:

$$N_{xx} = A_{11} \frac{\partial u}{\partial x} + (e_0 a)^2 \left(I_0 \frac{\partial^3 u}{\partial x \partial t^2} \right), \quad (23)$$

$$M_{xx} = D_{11} \frac{\partial^2 w}{\partial x^2} - (e_0 a)^2 \left(I_0 \frac{\partial^2 w}{\partial t^2} + q \right). \quad (24)$$

The governing equation of motion for the nanobeam can be derived from Eqs. (18), (19),

(23) and (24) as below:

$$A_{11} \frac{\partial^2 u}{\partial x^2} = I_0 \frac{\partial^2 u}{\partial t^2}, \quad (25)$$

$$D_{11} \frac{\partial^4 w}{\partial x^4} + I_0 \frac{\partial^2 w}{\partial t^2} - (e_0 a)^2 I_0 \frac{\partial^4 w}{\partial x^2 \partial t^2} = q - (e_0 a)^2 \frac{\partial^2 q}{\partial x^2}. \quad (26)$$

In this investigation, the transverse free vibration is considered, therefore, only the equation of motion (26) is used. Furthermore, neglecting the contribution of transverse distributed force, the equation of motion of the nanobeam becomes:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + I_0 \frac{\partial^2 w}{\partial t^2} - (e_0 a)^2 I_0 \frac{\partial^4 w}{\partial x^2 \partial t^2} = 0. \quad (27)$$

Equation (27) will be employed to investigate the free vibration of nanobeam with cracks in the following section.

The crossover flexibility constant represents the coupling between axial force and bending moment in the crack flexibility matrix. For slender beams undergoing transverse free vibration without axial pre-stress, the axial force contribution is negligible compared to bending effects. Moreover, fracture-mechanics-based evaluations show that the crossover flexibility constant is typically one to two orders of magnitude smaller than the bending-related flexibility constant for practical crack depth ratios. Since nonlocal elasticity modifies the constitutive relation at the continuum level but does not alter the geometric origin of crack flexibility coefficients, neglecting the crossover term remains a reasonable and widely adopted approximation for flexural vibration analysis of slender nanobeams.

3. Analytical solution

The equation of motion (27) can be analytically solved by using the separation-of-variables method. Accordingly, the solution of the equation of motion is assumed to have the following separated-variable form:

$$w(x, t) = W(x)e^{i\omega t}, \quad (28)$$

where, ω is the natural frequency of vibration. Substituting the solution (28) into Eq. (27), the following equation can be obtained:

$$\frac{d^4 W(x)}{dx^4} + (e_0 a)^2 \frac{I_0 \omega^2}{D_{11}} \frac{d^2 W(x)}{dx^2} - \frac{I_0 \omega^2}{D_{11}} W(x) = 0 \quad (29)$$

Introducing the dimension variables:

$$\bar{x} = \frac{x}{L}, \mu = \frac{e_0 a}{L}, \Lambda = \left(\frac{I_0 L^4}{D_{11}} \omega^2 \right)^{1/4}, \bar{W} = \frac{W}{L} \quad (30)$$

Employing Eq. (30), the equation of motion (29) can be rewritten in the following dimensionless form:

$$\frac{d^4 \bar{W}}{d\bar{x}^4} + \Lambda^4 \left(\mu^2 \frac{d^2 \bar{W}}{d\bar{x}^2} - \bar{W} \right) = 0. \quad (31)$$

The general solution of equation (31) can be determined in the following form:

$$\bar{W}(\bar{x}) = A_1 \sinh(\alpha_1 \bar{x}) + A_2 \cosh(\alpha_1 \bar{x}) + A_3 \sin(\alpha_2 \bar{x}) + A_4 \cos(\alpha_2 \bar{x}), \quad (32)$$

where, the coefficients α_1 and α_2 are defined as:

$$\alpha_1 = \mu \Lambda^2 \sqrt{\frac{\sqrt{1 + 4 / (\mu^4 \Lambda^4)} - 1}{2}}, \quad \alpha_2 = \mu \Lambda^2 \sqrt{\frac{\sqrt{1 + 4 / (\mu^4 \Lambda^4)} + 1}{2}} \quad (33)$$

Note that once the displacements are known (see Eq. (32)) and applying the definitions for the following quantities: the bending slope $\theta(\bar{x})$, the dimensionless bending moment $\bar{M}(\bar{x})$, and dimensionless shear force $\bar{S}(\bar{x})$, as below:

$$\begin{aligned} \theta(\bar{x}) &= \frac{dW(\bar{x})}{d\bar{x}}, \\ \bar{M}(\bar{x}) &= \frac{M(\bar{x})L}{EI} = \frac{d^2 W(\bar{x})}{d\bar{x}^2} + \mu^2 \Lambda^4 W(\bar{x}), \\ \bar{S}(\bar{x}) &= \frac{S(\bar{x})L^2}{EI} = \frac{d^3 W(\bar{x})}{d\bar{x}^3} + \mu^2 \Lambda^4 \frac{dW(\bar{x})}{d\bar{x}}. \end{aligned} \quad (34)$$

In the case without a crack, the four constants A_i ($i = 1 \div 4$) in the solution (Eq. (32)) are determined by solving a system of four equations derived from the appropriate boundary conditions at the nanobeam's supports, as follows:

Simply-supported end:

$$\bar{W} = 0, \bar{M} = \frac{d^2 \bar{W}}{d\bar{x}^2} + \mu^2 \Lambda^4 \bar{W} = 0, \quad (35)$$

Clamped end:

$$\bar{W} = 0, \frac{d\bar{W}}{d\bar{x}} = 0, \quad (36)$$

Free end:

$$\bar{M} = \frac{d^2 \bar{W}}{d\bar{x}^2} + \mu^2 \Lambda^4 \bar{W} = 0, \quad (37)$$

$$\bar{S} = \frac{d^3 \bar{W}}{d\bar{x}^3} + \mu^2 \Lambda^4 \frac{d\bar{W}}{d\bar{x}} = 0,$$

Let us consider that the nanobeam with an open crack at a distance $x = L_c$ from the left end.

Introducing the dimensionless variable $l_c = L_c / L, (0 \leq l_c \leq 1)$, which defines the location of the crack along the nanobeam. After establishing the modeling approach for the edge crack, we can proceed to analyze the free transverse vibration of the cracked nanobeam. The spatial solution for the transverse displacement in each of the two nanobeam segments can then be expressed in a form analogous to that of Eq. (31):

$$\frac{d^4 \bar{W}_1}{d\bar{x}^4} + \Lambda^4 \left(\mu^2 \frac{d^2 \bar{W}_1}{d\bar{x}^2} - \bar{W}_1 \right) = 0, \quad 0 \leq \bar{x} \leq l_c, \quad (38a)$$

$$\frac{d^4 \bar{W}_2}{d\bar{x}^4} + \Lambda^4 \left(\mu^2 \frac{d^2 \bar{W}_2}{d\bar{x}^2} - \bar{W}_2 \right) = 0, \quad l_c \leq \bar{x} \leq 1, \quad (38b)$$

Therefore, the general solution of the equation (38) can be expressed in the following form:

$$\begin{aligned} \bar{W}_1(\bar{x}) = & A_1 \sinh(\alpha_1 \bar{x}) + A_2 \cosh(\alpha_1 \bar{x}) \\ & + A_3 \sin(\alpha_2 \bar{x}) + A_4 \cos(\alpha_2 \bar{x}), \quad 0 \leq \bar{x} \leq l_c \end{aligned} \quad (39a)$$

$$\begin{aligned} \bar{W}_2(\bar{x}) = & A_5 \sinh(\alpha_1 \bar{x}) + A_6 \cosh(\alpha_1 \bar{x}) \\ & + A_7 \sin(\alpha_2 \bar{x}) + A_8 \cos(\alpha_2 \bar{x}), \quad l_c \leq \bar{x} \leq 1 \end{aligned} \quad (39b)$$

The general solutions (39) contain eight unknown constants $A_i (i=1 \div 8)$. These are obtained by enforcing both the support boundary conditions and the crack-location compatibility conditions. Specifically, four equations arise from the supports, while the remaining four come from

continuity requirements at the crack - ensuring seamless transverse displacement, bending slope, bending moment, and shear force across the crack. These conditions are outlined below:

Continuity of the transverse displacement:

$$\bar{W}_2(l_c) = \bar{W}_1(l_c). \quad (40)$$

Jump in the bending slope:

$$\bar{W}_2'(l_c) - \bar{W}_1'(l_c) = K \bar{W}_1(l_c). \quad (41)$$

Continuity of the bending moment:

$$\bar{W}_2''(l_c) + \mu^2 \Lambda^4 \bar{W}_2(l_c) = \bar{W}_1''(l_c) + \mu^2 \Lambda^4 \bar{W}_1(l_c). \quad (42)$$

Continuity of the shear force:

$$\bar{W}_2'''(l_c) + \mu^2 \Lambda^4 \bar{W}_2'(l_c) = \bar{W}_1'''(l_c) + \mu^2 \Lambda^4 \bar{W}_1'(l_c). \quad (43)$$

Consequently, the resulting formulation takes the form of an eight-equation algebraic system as shown below:

$$[H(\omega^2)] \{C\} = \{0\} \quad (44)$$

where, $[H(\omega^2)]$ is the dynamic stiffness matrix and $\{C\}$ is the mode shape vector. In the vibration analysis of structures, the dynamic stiffness matrix encapsulates both stiffness and inertia effects, forming a frequency-dependent system matrix. It governs the relationship between amplitude vectors and natural frequencies in the free vibration context. The mode shape vector $\{C\}$ contains coefficients representing the amplitude of each mode shape. These shapes describe how different portions of the structure move relative to one another at a natural frequency. A non-zero vector $\{C\}$ satisfying the homogeneous equation indicates a natural vibration mode, implying the system oscillates in those patterns independently of external excitation.

The presence of a nontrivial solution ($\{C\} \neq \{0\}$) means the determinant of $[H(\omega^2)]$ must be zero:

$$\det(H(\omega^2)) = 0 \quad (45)$$

This condition yields the system's natural

frequencies, while the corresponding vector $\{C\}$ defines the mode shapes.

In the case of a single crack, Eq. (44) forms a homogeneous system of eight algebraic equations. When additional cracks are present in the nanobeam, the number of equations increases

proportionally, depending on the number of cracks. Nevertheless, the solution procedure for a nanobeam with multiple cracks follows essentially the same approach as that for a single-crack scenario.

4. Numerical results

Table 1. Comparison of the first four-order dimensionless natural frequencies (Ω) of cracked nanobeams, with crack position $l_c = 0.25$

μ	Mode	K = 0		K = 0.065		K = 0.35		K = 2	
		Ref. [25]	Present	Ref. [25]	Present	Ref. [25]	Present	Ref. [25]	Present
Simply-Supported									
0.0	1	3.1416	3.1416	3.0921	3.0921	2.9071	2.9071	2.3493	2.3493
	2	6.2832	6.2832	6.1028	6.1028	5.6491	5.6491	5.1047	5.1047
	3	9.4248	9.4248	9.3021	9.3021	9.0767	9.0767	8.9008	8.9008
	4	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664
0.2	1	2.8908	2.8908	2.8447	2.8447	2.6645	2.6645	2.1134	2.1134
	2	4.9580	4.9580	4.8101	4.8101	4.4169	4.4169	3.9906	3.9906
	3	6.4520	6.4520	6.3638	6.3638	6.1924	6.1924	6.0692	6.0692
	4	7.6407	7.6407	7.6407	7.6407	7.6407	7.6407	7.6407	7.6407
0.4	1	2.4790	2.4790	2.4387	2.4387	2.2730	2.2730	1.7660	1.7660
	2	3.8203	3.8203	3.7047	3.7047	3.3993	3.3993	3.1123	3.1123
	3	4.7723	4.7723	4.7072	4.7072	4.5859	4.5859	4.5062	4.5062
	4	5.5509	5.5509	5.5509	5.5509	5.5509	5.5509	5.5509	5.5509
Clamped-Clamped									
0.0	1	4.7300	4.7300	4.6285	4.6285	4.3566	4.3566	3.9702	3.9702
	2	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532
	3	10.9956	10.9956	10.6976	10.6976	10.1028	10.1028	9.5833	9.5833
	4	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372
0.2	1	4.2766	4.2766	4.1736	4.1736	3.8855	3.8855	3.4764	3.4764
	2	6.0352	6.0352	6.0352	6.0352	6.0352	6.0352	6.0352	6.0352
	3	7.3840	7.3840	7.1421	7.1421	6.6089	6.6089	6.2045	6.2045
	4	8.4624	8.4624	8.4624	8.4624	8.4624	8.4624	8.4624	8.4624
0.4	1	3.5923	3.5923	3.4947	3.4947	3.2141	3.2141	2.6331	2.6331
	2	4.5978	4.5978	4.5978	4.5978	4.5978	4.5978	4.5978	4.5978
	3	5.4738	5.4738	5.2993	5.2993	4.9302	4.9302	4.6170	4.6170
	4	6.1504	6.1504	6.1504	6.1504	6.1504	6.1504	6.1504	6.1504

In this section, we present dimensionless natural frequencies $\Omega = \omega^2 L^4 \frac{\rho A}{EI}$, using the dimensionless nonlocal parameter $\mu = e_0 a / L$, and dimensionless crack location $l_c = L_c / L$. The cracked nanobeam is divided into segments at each crack, with boundary and continuity

conditions ensuring compatibility of transverse displacement $w(x, t)$, slope, bending M , and shear force S . Natural frequencies are obtained by solving the resulting homogeneous algebraic system via the determinant condition $\det(H(\omega^2)) = 0$, a standard dynamic stiffness-based eigenvalue approach. In this study, the

nanobeam material is assumed to be steel, characterized by the following properties:

Young's modulus: $E = 210 \times 10^9 (\text{Pa})$.

Mass density: $\rho = 7850 (\text{kg/m}^3)$.

Nanobeam thickness: $h = 20 (\text{nm})$ and width $b = 10 (\text{nm})$.

The first four-order dimensionless natural frequencies calculated in this study are compared with those reported by Loya et al. (2009) [25]. The comparison is conducted for a nanobeam

containing a single crack. These comparisons are summarized in Table 1 (crack position $l_c = 0.25$)

and in Table 2 (crack position $l_c = 0.5$). Our results exhibit exceptional agreement with those reported by Loya et al. (2009) [25], reinforcing the accuracy and validity of our current model.

In the following section, we conduct a comprehensive parametric study to examine how key variables influence the free vibration behavior of multi-cracked nanobeams. The parameters of interest include:

Table 2. Comparison of the first four-order dimensionless natural frequencies (Ω) of cracked nanobeams, with crack position $l_c = 0.5$

μ	Mode	K = 0		K = 0.065		K = 0.35		K = 2	
		Ref. [25]	Present	Ref. [25]	Present	Ref. [25]	Present	Ref. [25]	Present
Simply-Supported									
0.0	1	3.1416	3.1416	3.0469	3.0469	2.7496	2.7496	2.0960	2.0960
	2	6.2832	6.2832	6.2832	6.2832	6.2832	6.2832	6.2832	6.2832
	3	9.4248	9.4248	9.1669	9.1669	8.6129	8.6129	8.0730	8.0730
	4	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664
0.2	1	2.8908	2.8908	2.8031	2.8031	2.5233	2.5233	1.9098	1.9098
	2	4.9581	4.9581	4.9581	4.9581	4.9581	4.9581	4.9581	4.9581
	3	6.4520	6.4520	6.2604	6.2604	5.7891	5.7891	5.3416	5.3416
	4	7.6407	7.6407	7.6407	7.6407	7.6407	7.6407	7.6407	7.6407
0.4	1	2.4790	2.4790	2.4032	2.4032	2.1567	2.1567	1.6195	1.6195
	2	3.8204	3.8204	3.8204	3.8204	3.8204	3.8204	3.8204	3.8204
	3	4.7723	4.7723	4.6284	4.6284	4.2729	4.2729	3.9563	3.9563
	4	5.5509	5.5509	5.5509	5.5509	5.5509	5.5509	5.5509	5.5509
Clamped-Clamped									
0.0	1	4.7300	4.7300	4.6285	4.6285	4.3566	4.3566	3.9702	3.9702
	2	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532
	3	10.9956	10.9956	10.6976	10.6976	10.1028	10.1028	9.5833	9.5833
	4	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372
0.2	1	4.2766	4.2766	4.1736	4.1736	3.8855	3.8855	3.4764	3.4764
	2	6.0352	6.0352	6.0352	6.0352	6.0352	6.0352	6.0352	6.0352
	3	7.3840	7.3840	7.1421	7.1421	6.6089	6.6089	6.2045	6.2045
	4	8.4624	8.4624	8.4624	8.4624	8.4624	8.4624	8.4624	8.4624
0.4	1	3.5923	3.5923	3.4947	3.4947	3.2141	3.2141	2.6331	2.6331
	2	4.5978	4.5978	4.5978	4.5978	4.5978	4.5978	4.5978	4.5978
	3	5.4738	5.4738	5.2993	5.2993	4.9302	4.9302	4.6170	4.6170
	4	6.1504	6.1504	6.1504	6.1504	6.1504	6.1504	6.1504	6.1504

Nonlocal parameter, $\mu = e_0 a / L$: encapsulating size-dependent, long-range elastic

interactions per Eringen's nonlocal theory.

Crack-severity, K: measuring the compliance

or flexibility introduced by each crack.

Boundary conditions (e.g., Clamped–

Clamped, Simply-Supported, Clamped-Free, and Clamped-Simply Supported).

Table 3. Dimensionless natural frequencies (Ω) of multi-cracked nanobeams, with S-S boundary conditions

μ	Mode	K = 0	K = 0.5	K = 1.0	K = 1.5	K = 2.0
0	1	3.1416	2.3869	2.1007	1.9312	1.8136
	2	6.2832	4.7664	4.1920	3.8524	3.6171
	3	9.4248	7.0471	6.1638	5.6504	5.2977
0.1	1	3.0685	2.3302	2.0503	1.8847	1.7698
	2	5.7817	4.3556	3.8197	3.5056	3.2889
	3	8.0400	5.8552	5.0806	4.6418	4.3441
0.2	1	2.8908	2.1928	1.9285	1.7722	1.6640
	2	4.9581	3.7048	3.2390	2.9685	2.7828
	3	6.4520	4.6186	3.9917	3.6414	3.4051
0.3	1	2.6800	2.0306	1.7850	1.6400	1.5396
	2	4.3013	3.2007	2.7942	2.5592	2.3982
	3	5.4422	3.8726	3.3431	3.0483	2.8498
0.4	1	2.4790	1.8768	1.6492	1.5150	1.4221
	2	3.8204	2.8370	2.4749	2.2660	2.1232
	3	4.7722	3.3874	2.9229	2.6647	2.4910
0.5	1	2.3022	1.7419	1.5303	1.4056	1.3194
	2	3.4604	2.5668	2.2384	2.0492	1.9198
	3	4.2941	3.0442	2.6262	2.3940	2.2378

Number of Cracks.

The dimensionless nonlocal parameter μ is defined based on the intrinsic material length scale parameter of Eringen's theory and the beam length. The selected range of μ is consistent with commonly reported values for nanoscale beams in the literature, allowing the investigation of weak to moderate size-dependent effects while maintaining physical relevance.

In the present analysis, we consider a nanobeam containing three cracks located at the normalized positions $l_c = 0.25$, $l_c = 0.5$ and $l_c = 0.75$. The crack severities are assumed identical for all cracks, i.e. $K_1 = K_2 = K_3 = K$. The dimensionless natural frequencies of the nanobeam under different boundary conditions are listed in Tables 3–5. In this investigation, four types of boundary conditions are considered including

simply-supported (S–S), clamped–clamped (C–C), clamped–free (C–F), and clamped – simply supported (C–S) boundary conditions. Table 3 reports the dimensionless natural frequencies (Ω) of the three-cracked nanobeam with S–S boundary conditions for various values of the crack-severity parameter K and dimensionless nonlocal parameter μ .

Table 4 presents the dimensionless natural frequencies (Ω) of the three-cracked nanobeam under C–C boundary conditions, as functions of the crack-severity parameter K , and the dimensionless nonlocal parameter μ .

Table 5 displays the dimensionless natural frequencies (Ω) of the three-cracked nanobeam under C–F boundary conditions, for various values of the crack-severity parameter K and the dimensionless nonlocal parameter μ .

Table 4. Dimensionless natural frequencies (Ω) of multi-cracked nanobeams, with C-C boundary conditions

μ	Mode	K = 0	K = 0.5	K = 1.0	K = 1.5	K = 2.0
0	1	4.7300	4.2489	3.9935	3.8035	3.6513
	2	7.8532	6.7792	6.5035	6.3757	6.3018
	3	10.9956	8.2683	7.5581	7.2205	7.0230
0.1	1	4.5945	4.0920	3.8090	3.5989	3.4336
	2	7.1402	6.1013	5.8291	5.7038	5.6316
	3	9.2583	6.7606	6.1939	5.9468	5.8105
0.2	1	4.2766	3.7329	3.4061	3.1754	3.0030
	2	6.0352	5.1017	4.8540	4.7410	4.6762
	3	7.3840	5.3776	5.0075	4.8653	4.7921
0.3	1	3.9184	3.3501	3.0099	2.7841	2.6214
	2	5.1963	4.3707	4.1508	4.0510	3.9940
	3	6.2317	4.5922	4.3360	4.2433	4.1964
0.4	1	3.5923	3.0240	2.6934	2.4821	2.3325
	2	4.5978	3.8584	3.6611	3.5719	3.5209
	3	5.4738	4.0785	3.8813	3.8111	3.7754
0.5	1	3.3153	2.7618	2.4480	2.2518	2.1141
	2	4.1561	3.4835	3.3040	3.2228	3.1765
	3	4.9328	3.7046	3.5406	3.4821	3.4523

Table 5. Dimensionless natural frequencies (Ω) of multi-cracked nanobeams, with C-F boundary conditions

μ	Mode	K = 0	K = 0.5	K = 1.0	K = 1.5	K = 2.0
0	1	1.8751	1.5535	1.3966	1.2964	1.2241
	2	4.6941	3.8652	3.5003	3.2642	3.0917
	3	7.8548	6.3238	5.7479	5.3739	5.0949
0.1	1	1.8792	1.5575	1.4002	1.2996	1.2272
	2	4.5475	3.7186	3.3514	3.1161	2.9455
	3	7.1459	5.6252	5.0032	4.6110	4.3328
0.2	1	1.8919	1.5699	1.4115	1.3101	1.2371
	2	4.1924	3.3752	3.0152	2.7896	2.6288
	3	6.0674	4.6382	4.0408	3.6931	3.4561
0.3	1	1.9154	1.5933	1.4328	1.3299	1.2557
	2	3.7665	2.9835	2.6485	2.4430	2.2982
	3	5.2988	3.9767	3.4314	3.1272	2.9226
0.4	1	1.9543	1.6342	1.4703	1.3648	1.2886
	2	3.3456	2.6114	2.3106	2.1286	2.0012
	3	4.8370	3.5523	3.0402	2.7651	2.5818
0.5	1	2.0219	1.7159	1.5479	1.4380	1.3582
	2	2.9433	2.2513	1.9853	1.8271	1.7169
	3	4.5348	3.2681	2.7687	2.5133	2.3447

Table 6 presents the dimensionless natural frequencies (Ω) of the three-cracked nanobeam under C-S boundary conditions, as functions of the crack-severity parameter K, and the dimensionless

nonlocal parameter μ .

It is evident that as the dimensionless nonlocal parameter μ increases, the natural frequencies of the nanobeam systematically decrease. This trend is both intuitively reasonable and consistent with existing results in nonlocal elasticity theory. The nonlocal parameter μ serves to introduce size-dependent stiffening - or more precisely, size-dependent softening - into continuum models. Larger values of μ effectively increase the influence of long-range atomic interactions, which manifests in a reduction of the nanobeam's apparent stiffness. Consequently, the structural system becomes "softer," and its natural frequencies diminish. In the classical framework of nonlocal Euler–Bernoulli beam theory, an increase in nonlocal parameters leads to a consistent softening of natural frequencies for various boundary conditions (clamped–clamped, simply supported, clamped-free, clamped – simply supported). In designing NEMS, the proper selection of the nonlocal parameter μ is crucial, as it significantly influences the device's dynamic behavior, accuracy, and performance. Based on both theoretical insights and recent literature, here are key recommendations as follow. Different nonlocal formulations yield contrasting effects. These results guideline aim to help engineer choose the nonlocal parameter μ judiciously to ensure robust, predictable, and physically sound NEMS designs.

A brief discussion regarding the so-called nonlocal paradox is necessary in the context of the adopted differential formulation of Eringen's theory. It has been reported in the literature that, for certain boundary conditions—particularly the cantilever (C–F) configuration—the differential form of nonlocal elasticity may yield responses that deviate from those predicted by the integral form. In some studies, this discrepancy manifests as an artificial hardening effect or an absence of the expected frequency reduction with increasing nonlocal parameter. It is important to emphasize that this

paradox is not an intrinsic deficiency of the differential model itself, but rather a consequence of inconsistent implementation of nonlocal boundary conditions. When classical (local) boundary conditions are directly imposed on nonlocal stress resultants without accounting for the constitutive relation between stress and strain fields, inconsistencies may arise. In the present study, the governing equations are derived systematically within the differential nonlocal framework, and the associated boundary conditions are enforced in a manner fully consistent with the adopted constitutive model. As demonstrated in the numerical results, the natural frequencies decrease monotonically with increasing dimensionless nonlocal parameter μ for all boundary conditions considered, including the cantilever case. This physically consistent softening behavior confirms that the present formulation does not suffer from the commonly reported paradox. Although the integral formulation provides a more fundamental nonlocal representation, its application to multi-cracked nanobeams involving multiple internal compatibility conditions significantly increases mathematical complexity. The differential approach therefore represents a computationally efficient and theoretically consistent alternative for the present problem.

The obtained results (from Tables 3-6) clearly show that, for all four boundary conditions considered (S-S, C-C, C-F, and C-S), the natural frequencies (Ω) of the nanobeam consistently decrease as the crack severity K increases. This trend aligns with common findings in literature: increasing the crack depth or length leads to a reduction in stiffness, thereby lowering the natural frequencies. Specifically: for multiple cracked nanobeams under various boundary conditions, studies using stress-driven nonlocal models confirm that all natural frequencies drop as the crack length grows [26]. In functionally graded, double-cracked nanobeams, a recent differential

quadrature study similarly reports that increased crack severity leads to reduced fundamental frequencies across boundary types [13]. Even in nonlinear vibration analyses of microbeams, frequency decreases markedly with increasing crack severity, particularly for vibrations under

nonlinear regimes [28]. Thus, the present findings are fully consistent with established theoretical and numerical studies: as crack severity K increases, the structural stiffness diminishes, leading to lower natural frequencies under S-S, C-C, C-F, and C-S conditions alike.

Table 6. Dimensionless natural frequencies (Ω) of multi-cracked nanobeams, with C-S boundary conditions

μ	Mode	$K = 0$	$K = 0.5$	$K = 1.0$	$K = 1.5$	$K = 2.0$
0	1	3.9266	3.2415	2.9304	2.7295	2.5833
	2	7.0686	5.7580	5.2402	4.9034	4.6531
	3	10.2102	7.8648	7.2081	6.9064	6.7362
0.1	1	3.8209	3.1379	2.8278	2.6291	2.4852
	2	6.4649	5.1779	4.6373	4.2927	4.0449
	3	8.6517	6.5517	6.0505	5.8421	5.7299
0.2	1	3.5701	2.9000	2.5977	2.4070	2.2705
	2	5.5079	4.3125	3.7996	3.4893	3.2736
	3	6.9204	5.2686	4.9372	4.8060	4.7360
0.3	1	3.2828	2.6396	2.3528	2.1746	2.0481
	2	4.7668	3.6846	3.2251	2.9541	2.7678
	3	5.8371	4.4961	4.2483	4.1510	4.0990
0.4	1	3.0176	2.4085	2.1401	1.9749	1.8583
	2	4.2312	3.2486	2.8354	2.5945	2.4297
	3	5.1213	3.9799	3.7771	3.6976	3.6551
0.5	1	2.7899	2.2158	1.9650	1.8116	1.7038
	2	3.8325	2.9312	2.5547	2.3365	2.1875
	3	4.6105	3.6045	3.4293	3.3606	3.3239

Fig. 2 illustrates how the first dimensionless natural frequency (Ω) decreases as the dimensionless nonlocal parameter (μ) increases, for various levels of crack severity (K). The variation of the natural frequency with respect to the dimensionless nonlocal parameter depends significantly on the boundary conditions. For simply - supported (S-S), clamped - clamped (C-C), and clamped - simply supported (C-S) boundary conditions, the natural frequency decreases as the nonlocal parameter increases. This reflects the “softening” effect introduced by the nonlocal elasticity model, i.e. the effective stiffness of the beam is reduced by the size-scale effects and long-range interactions. However, for the clamped - free (C-F or cantilever) boundary case, an interesting

behavior is observed: the fundamental natural frequency increases slightly when the nonlocal parameter is small (e.g. $\mu \leq 0.6$), and beyond that threshold it begins to decrease as the nonlocal parameter μ continues to grow. This initial increase can be linked to peculiarities in the nonlocal model for cantilever beams, especially relating to how free-end boundary conditions are implemented and whether the model remains well-posed. Some works in the literature report similar behavior: a slight “stiffening” effect appears for small values of the nonlocal parameter in cantilever configurations, but this is often reversed for larger nonlocal parameter values [30].

The cantilever (C-F) beam exhibits a non-monotonic variation of the fundamental frequency

with respect to the nonlocal parameter μ . Specifically, a slight increase in frequency is observed when μ is small, followed by a decreasing trend as μ increases further. This behavior differs from the monotonic softening observed in S-S, C-C, and C-S configurations. This phenomenon can be interpreted as a competition between localized boundary effects and global nonlocal softening. In a cantilever beam, bending stresses are strongly concentrated near the clamped end, producing steep stress gradients. For small μ values, the differential nonlocal constitutive relation mainly induces local stress redistribution rather than a dominant global stiffness reduction. Due to the asymmetric boundary constraint, this redistribution may slightly enhance the effective bending resistance near the fixed support, leading to a small increase in the natural frequency. As μ increases, long-range interatomic interactions progressively influence the entire beam domain. The resulting reduction in effective bending stiffness then becomes dominant, and the classical softening behavior of nonlocal elasticity emerges. Therefore, the observed transition from slight stiffening to clear softening reflects the interplay between boundary-induced stress concentration and size-dependent nonlocal effects.

Fig. 3 demonstrates that, for all examined boundary conditions (S-S, C-C, C-S and C-F), the dimensionless natural frequencies - specifically Mode 1 and Mode 3 - decrease as crack severity K increases. This figure is plotted with a specific value of the dimensionless nonlocal parameter μ . Some key observations from the obtained results as follows. Monotonic decline in the natural frequency: For each case, the natural frequency drops as K increases, reflecting the loss of stiffness due to more severe cracks. Variation across boundary conditions: the C-C case starts with the highest natural frequencies and shows the least sensitivity to increasing crack severity K . Examining the plot

shows that the S-S case exhibits the steepest decline in dimensionless natural frequency (Mode 1) as crack severity K increases, compared with the C-C, C-S and C-F cases. This indicates that, for the considered modes and parameter range, the S-S configuration is the most sensitive to crack-induced loss of stiffness. The boundary-condition dependence of frequency reduction (and its mode/location sensitivity) is well documented in the literature [16-28]. Mode-dependent sensitivity: Mode 3 frequencies consistently lie above Mode 1 but follow a similar downward trend, suggesting that higher modes are also softened by crack severity, albeit maintaining higher baseline frequencies. This behavior aligns with established findings in the literature: cracks reduce structural stiffness and consequently lower natural frequencies. For example, Sun et al. (2024) [28] emphasized that crack-induced frequency reductions are strongly dependent on both crack location and severity and are notably more pronounced in lower modes or under flexible boundary conditions. Additionally, earlier studies confirm that as crack severity increases, natural frequencies decrease consistently across different beam configurations. While it is expected that increasing crack severity reduces the natural frequencies due to stiffness degradation, the nonlocal framework reveals more subtle mechanisms related to crack shielding and amplification effects. In the present multi-cracked nanobeam model, each crack is represented by a localized rotational flexibility, which introduces additional compliance into the structure. In the classical (local) elasticity framework, the reduction in stiffness is purely governed by crack depth and location. However, in the nonlocal model, long-range interactions redistribute stresses over a finite internal length scale. For moderate to large values of the nonlocal parameter μ , the stress field becomes less localized around the crack tip. This redistribution can partially reduce the effective stress concentration at the crack location, thereby

mitigating the influence of the crack on global stiffness. Such a mechanism may be interpreted as a crack shielding effect induced by nonlocal interactions. Conversely, when μ is small, the nonlocal interaction length is limited, and the crack-induced local flexibility remains dominant. In certain boundary configurations, especially those involving strong stress gradients, the interaction between nonlocal stress redistribution and crack compliance may enhance the sensitivity of the structure to crack severity. This behavior can be viewed as a crack amplification effect. Therefore, the dynamic response of multi-cracked nanobeams in the nonlocal framework is governed by the competition between: localized stiffness degradation due to cracks, and global stress redistribution associated with long-range nonlocal interactions. This interaction explains why the influence of crack severity is not purely linear and may vary depending on both μ and boundary conditions.

Fig. 4 illustrates that increasing the number of cracks in the nanobeams decreases in the dimensionless natural frequencies for both Mode 1 and Mode 3 across all boundary conditions (S-S, C-C, C-S and C-F). In this investigation, the crack-severity is considered as $K = 0.5$ and the dimensionless nonlocal parameter is selected as $\mu = 0.2$. It can be observed that both Mode 1 and Mode 3 frequencies decrease steadily as cracks accumulate. The frequency difference between Mode 1 and Mode 3 is maintained across all crack counts, indicating mode shapes shift similarly under the influence of cracking. Among boundary conditions, the C-C configuration displays the steepest decline, suggesting that beams with both ends fixed are most vulnerable to stiffness loss due to multiple cracks. This behavior aligns with analytical and numerical findings in the literature, where researchers demonstrate that adding cracks consistently reduces natural frequencies, with the degree of reduction depending on boundary

constraints and mode number [29]. In the multi-crack analysis, the cracks are assumed to be equally spaced and to have identical severity. This configuration provides a controlled reference case that enables systematic evaluation of crack–nonlocal interaction effects. The formulation itself is not restricted to uniform crack distributions, and arbitrary crack locations and severities can be incorporated without modification of the governing equations.

Fig. 5 illustrates the first three normalized mode shapes of multi-cracked nanobeams under four classical boundary conditions: S–S, C–C, C–S and C–F. In this examination, nanobeams with three cracks at positions $l_c = 0.25$, $l_c = 0.5$ and $l_c = 0.75$ are considered. The dimensionless nonlocal parameter is considered as $\mu = 0.2$ and the crack-severity parameter is chosen as $K_1 = K_2 = K_3 = 0.5$. It can be observed that:

S–S boundary conditions: Mode 1 exhibits a single curvature with zero displacement at the supports and a center anti-node. Mode 2 shows two half-waves with a central node near midspan and symmetric peaks. Mode 3 introduces a more complex pattern with three half-waves, showing alternate positive and negative lobes between the supports.

C–C boundary conditions: Mode 1 displays zero slope and zero displacement at both ends, leading to a central bulge with symmetric curvature. Mode 2 forms two lobes with an internal node, and Mode 3 consists of three lobes, each separated by internal nodes, manifesting higher spatial complexity due to the fully restrained ends.

C–F boundary conditions: Mode 1 rises monotonically from the fixed end to the free end, featuring maximum displacement - and zero slope - at the free end. Mode 2 shows a single node near the beam's middle, with a lobe that peaks closer to the free end. Mode 3 introduces two internal nodes, resulting in more sign changes and a richer shape complexity compared to Mode 1 and Mode 2.

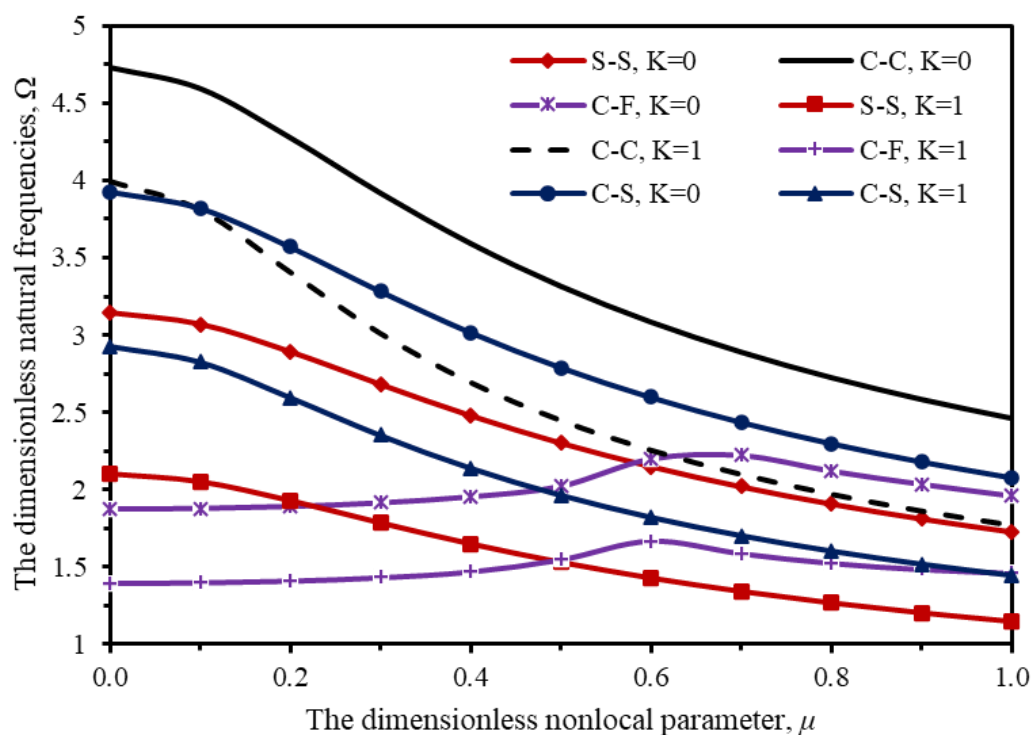


Fig. 2. The variation of the first dimensionless natural frequency (Ω) of multi-cracked nanobeams to the dimensionless nonlocal parameter (μ)

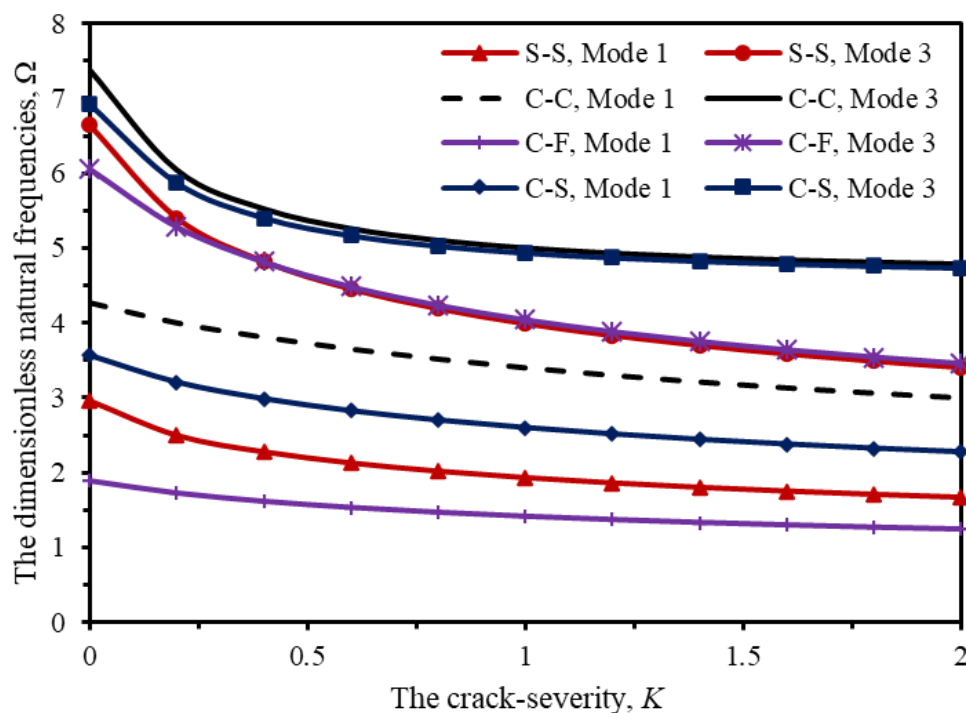


Fig. 3. The variation of the dimensionless natural frequencies (Ω_1 and Ω_3) of multi-cracked nanobeams to the crack-severity parameter (K)

C-S boundary conditions: Mode 1 starts with zero displacement and zero slope at the clamped end, rising towards the simply supported end, achieving maximum displacement (anti-node) near

the simply supported side. Mode 2 exhibits an internal node somewhere between clamped and supported ends. It has two lobes: one from the clamped end to the node, and the other from the

node to the support, with larger amplitude near the supported end. Mode 3 introduces two internal nodes, resulting in three alternating lobes of displacement (positive and negative). The complexity increases with mode number, and the lobes near the simply supported end generally show higher amplitude.

Overall, these mode shapes reflect how the combination of multiple cracks and different

support conditions significantly alter the vibrational profiles of Euler–Bernoulli nanobeams. Multiple cracks act as rotational discontinuities, influencing curvature and shift nodal positions in a manner that is highly sensitive to the applied boundary conditions. This pattern highlights the importance of boundary constraints when interpreting mode shapes for crack detection, frequency analysis, or the design of nano-scale beam resonators.

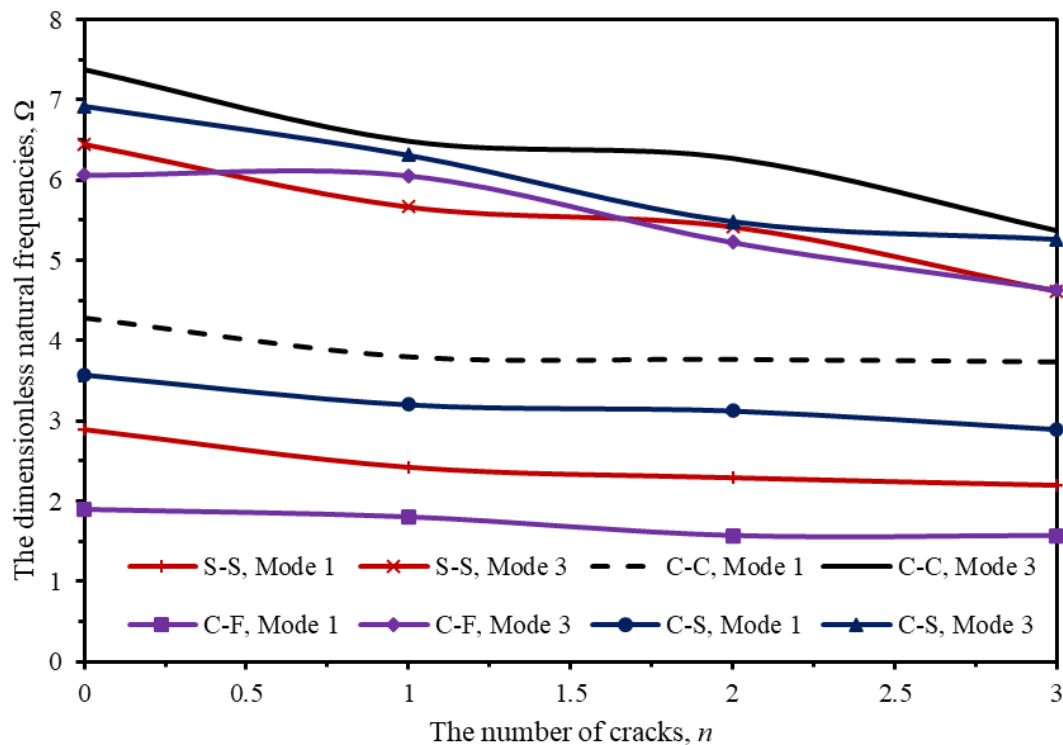


Fig. 4. The variation of the dimensionless natural frequencies (Ω_1 and Ω_3) of cracked nanobeams to the number of cracks (n)

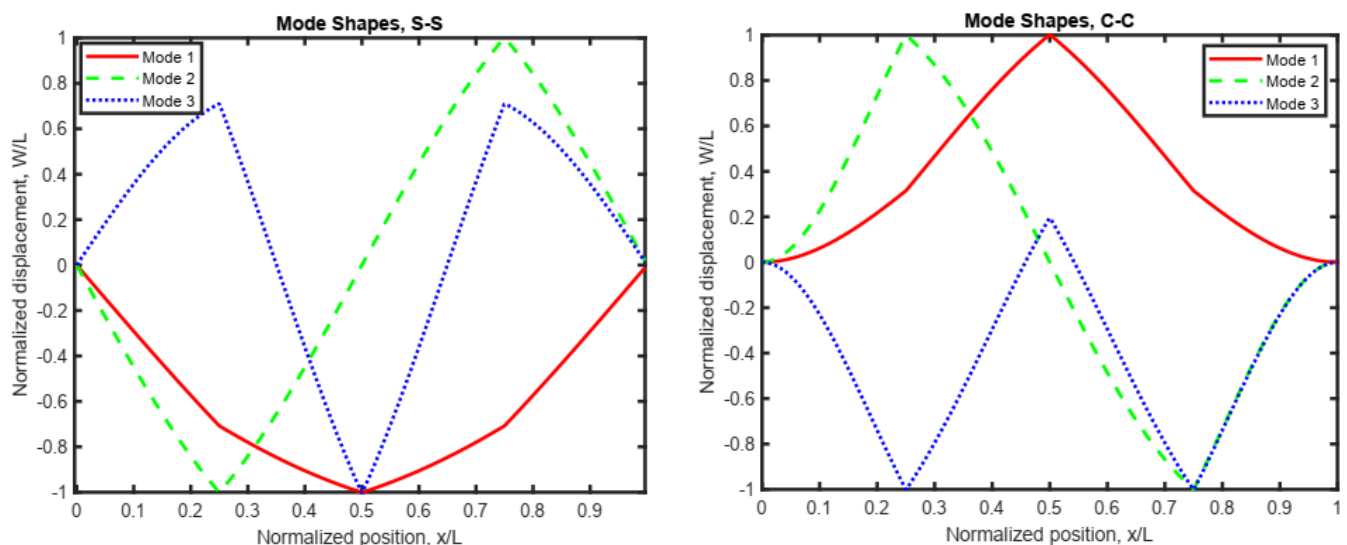


Fig. 5. Mode shapes of multi-cracked nanobeams

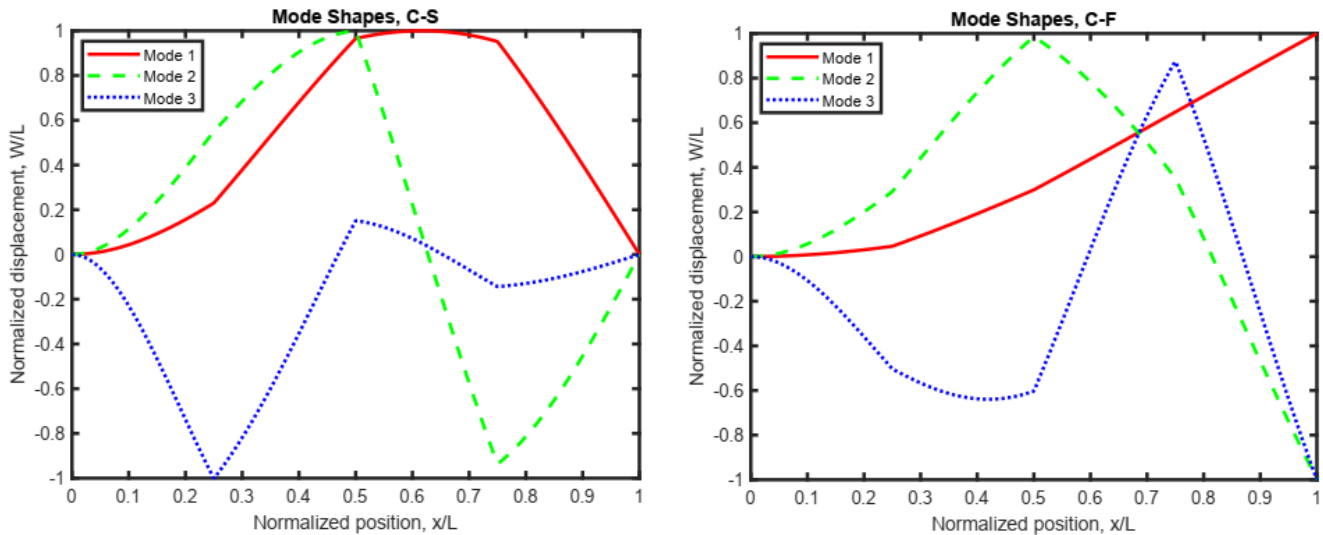


Fig. 5. (continued)

5. Conclusions

This study has developed an analytical model to examine the free transverse vibrations of multi-cracked nanobeams, based on the Euler–Bernoulli beam theory enhanced by Eringen’s nonlocal elasticity. By modeling cracks as rotational springs, the analysis encapsulates discontinuities in slope and displacement and derives natural frequencies and mode shapes through rigorous enforcement of kinematic and continuity conditions across crack positions and boundaries.

Key findings:

The presence of multiple cracks significantly reduces natural frequencies compared to intact or local-behavior predictions.

The nonlocal parameter and crack severity both exacerbate frequency reduction, highlighting pronounced size-dependent effects at the nanoscale.

The method's generality enables application to arbitrary crack configurations and boundary conditions, making it a versatile analytical tool.

Implications:

Offers benchmark solutions for validating numerical and experimental studies in nanobeam vibration.

Supports crack detection, structural health monitoring, and the design optimization of nano-

resonators where precision in vibrational behavior is critical.

In summary, the proposed analytical framework captures essential nonlocal and discontinuity effects in cracked nanobeams, delivering accurate predictions of vibrational characteristics and establishing a dependable reference point for future theoretical, numerical, and experimental investigations.

The present formulation is general and material-independent, and thus can be directly extended to nanostructures composed of advanced materials such as carbon nanotubes, graphene, or semiconductor nanowires by incorporating the corresponding material constants. Such extensions would further enhance the applicability of the proposed model to realistic NEMS devices and will be considered in future investigations.

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Declaration of Generative AI

During the preparation of this work, the authors do not use Generative AI and AI-assisted technologies tools in order to generate the content of the manuscript.

References

- [1] K.L. Ekinci, M.L. Roukes. (2005).

- Nanoelectromechanical systems. *Review of Scientific Instruments*, 76, 061101. <https://doi.org/10.1063/1.1927327>
- [2] B. Xu, P. Zhang, J. Zhu, et al. (2022). Nanomechanical Resonators: Toward Atomic Scale. *ACS Nano*, 16(10), 15545–15585. <https://doi.org/10.1021/acsnano.2c01673>
- [3] K. Eom, H.S. Park, D.S. Yoon, T. Kwon. (2011). Nanomechanical resonators and their applications in biological/chemical detection: Nanomechanics principles. *Physics Reports*, 503(4–5), 115–163, <https://doi.org/10.1016/j.physrep.2011.03.002>
- [4] M.C. Lemme, S. Wagner, K. Lee, et al. (2020). Nanoelectromechanical Sensors based on Suspended 2D Materials. *Research*, 2020, 8748602. <https://doi.org/10.34133/2020/8748602>
- [5] A.C. Eringen. (1972). Nonlocal polar elastic continua. *International Journal of Engineering Science*, 10(1), 1–16. [https://doi.org/10.1016/0020-7225\(72\)90070-5](https://doi.org/10.1016/0020-7225(72)90070-5)
- [6] A.C. Eringen. (1972). Linear theory of nonlocal elasticity and dispersion of plane waves. *International Journal of Engineering Science*, 10(5), 425–435. [https://doi.org/10.1016/0020-7225\(72\)90050-X](https://doi.org/10.1016/0020-7225(72)90050-X)
- [7] B. Arash, Q. Wang. (2012). A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes. *Computational Materials Science*, 51(1), 303–313. <https://doi.org/10.1016/j.commatsci.2011.07.040>
- [8] C.-P. Wu, J.-J. Yu. (2019). A review of mechanical analyses of rectangular nanobeams and single-, double-, and multi-walled carbon nanotubes using Eringen's nonlocal elasticity theory. *Archive of Applied Mechanics*, 89, 1761–1792. <https://doi.org/10.1007/s00419-019-01542-z>
- [9] B.N. Novakovic, B. Caric. (2024). A nonlocal beam with nonsymmetrical boundary conditions: stability analysis and shape optimization. *Archive of Applied Mechanics*, 94, 1263–1272. <https://doi.org/10.1007/s00419-024-02573-x>
- [10] R.M. Mamu, N. Togun. (2024). Thermal effects on nonlinear vibration of nonlocal nanobeam embedded in nonlinear elastic medium. *Acta Mechanica*, 235, 3483–3512. <https://doi.org/10.1007/s00707-024-03894-2>
- [11] A. Amendola, V. Zampoli, R. Luciano. (2023). Damped waves under nonlocal Euler–Bernoulli and extended Green–Naghdi II theories in radiating thermoelastic nanobeams. *Acta Mechanica*, 234, 2077–2085. <https://doi.org/10.1007/s00707-023-03478-6>
- [12] M. Hossain, J. Lellep. (2023). Natural Vibration of Axially Graded Multi-cracked Nanobeams in Thermal Environment Using Power Series. *Journal of Vibration Engineering & Technologies*, 11, 1–18. <https://doi.org/10.1007/s42417-022-00555-3>
- [13] M.A. Attia, M.S. Matbuly, T. Osman, M. AbdElkhalek. (2024). Dynamic analysis of double cracked bi-directional functionally graded nanobeam using the differential quadrature method. *Acta Mechanica*, 235, 1961–2012. <https://doi.org/10.1007/s00707-023-03797-8>
- [14] M.S. Taima, T.A. El-Sayed, M.I. Friswell. (2023). Thermal vibration analysis of cracked nanobeams submerged in elastic foundations by nonlocal continuum mechanics. *Thin-Walled Structures*, 193, 111249. <https://doi.org/10.1016/j.tws.2023.111249>
- [15] M. Eghbali, S.A. Hosseini, M. Pourseifi. (2023). An dynamical evaluation of size-dependent weakened nano-beam based on the nonlocal strain gradient theory. *The Journal of Strain Analysis for Engineering Design*, 58(5), 357–366. <https://doi.org/10.1177/03093247221135210>
- [16] D. Scorza, R. Luciano, A. Carpinteri, S. Vantadori. (2025). A novel analytical approach

- for simulating the mechanical behavior of multi-cracked nanobeams. *Engineering Fracture Mechanics*, 326, 111353. <https://doi.org/10.1016/j.engfracmech.2025.111353>
- [17] D. Scorza, A. Carpinteri, R. Luciano, C. Ronchei, S. Vantadori, A. Zanichelli. (2024). Mechanical Behaviour of Multiple Cracked Nanobeams: A Novel Analytical Model. *Procedia Structural Integrity*, 66, 406-411. <https://doi.org/10.1016/j.prostr.2024.11.092>
- [18] G. Lovisi. (2023). Application of the surface stress-driven nonlocal theory of elasticity for the study of the bending response of FG cracked nanobeams. *Composite Structures*, 324, 117549. <https://doi.org/10.1016/j.compstruct.2023.117549>
- [19] A. Caporale, R. Luciano, D. Scorza, S. Vantadori. (2023). Local–nonlocal stress-driven model for multi-cracked nanobeams. *International Journal of Solids and Structures*, 273, 112230. <https://doi.org/10.1016/j.ijsolstr.2023.112230>
- [20] H. Qing, Y. Tang. (2023). Size-dependent fracture analysis of Centrally-Cracked nanobeam using Stress-Driven Two-Phase Local/Nonlocal integral model with discontinuity and symmetrical conditions. *Engineering Fracture Mechanics*, 282, 109193. <https://doi.org/10.1016/j.engfracmech.2023.109193>
- [21] A.I. Aria, M.I. Friswell, T. Rabczuk. (2019). Thermal vibration analysis of cracked nanobeams embedded in an elastic matrix using finite element analysis. *Composite Structures*, 212, 118-128. <https://doi.org/10.1016/j.compstruct.2019.01.040>
- [22] A. Bahrami. (2017). A wave-based computational method for free vibration, wave power transmission and reflection in multi-cracked nanobeams. *Composites Part B: Engineering*, 120, 168-181. <https://doi.org/10.1016/j.compositesb.2017.03.053>
- [23] M. Loghmani, M.R.H. Yazdi. (2018). An analytical method for free vibration of multi cracked and stepped nonlocal nanobeams based on wave approach. *Results in Physics*, 11, 166-181. <https://doi.org/10.1016/j.rinp.2018.08.046>
- [24] S. Hosseini-Hashemi, M. Fakher, R. Nazemnezhad, M.H.S. Haghighi. (2014). Dynamic behavior of thin and thick cracked nanobeams incorporating surface effects. *Composites Part B: Engineering*, 61, 66-72. <https://doi.org/10.1016/j.compositesb.2014.01.031>
- [25] J. Loya, J. López-Puente, R. Zaera, J. Fernández-Sáez. (2009). Free transverse vibrations of cracked nanobeams using a nonlocal elasticity model. *Journal of Applied Physics*, 105(4), 044309. <https://doi.org/10.1063/1.3068370>
- [26] H. Darban, R. Luciano, M. Basista. (2022). Free transverse vibrations of nanobeams with multiple cracks. *International Journal of Engineering Science*, 177, 103703. <https://doi.org/10.1016/j.ijengsci.2022.103703>
- [27] D. Atcı. (2025). Investigation of Crack Effects on the Nonlinear Vibrations of Microbeams with a Tip Mass in a Magnetic Field. *Dokuz Eylül Üniversitesi Mühendislik Fakültesi Fen ve Mühendislik Dergisi*, 27(79), 54-61. <https://doi.org/10.21205/deufmd.2025277908>
- [28] X. Sun, S. Ilanko, Y. Mochida, R.C. Tighe. (2024). A Study on the Feasibility of Natural Frequency-Based Crack Detection. *Applied Sciences*, 14(24), 11712. <https://doi.org/10.3390/app142411712>
- [29] A. Rahi, H. Petoft. (2018). Free vibration analysis of multi-cracked micro beams based on Modified Couple Stress Theory. *Journal of Theoretical and Applied Vibration and Acoustics*, 4(2), 205-222. doi

10.22064/tava.2019.89997.1113

- [30] N. Challamel, Z. Zhang, C.M. Wang, et al. (2014). On nonconservativeness of Eringen's nonlocal elasticity in beam mechanics:

correction from a discrete-based approach. *Archive of Applied Mechanics*, 84, 1275–1292. <https://doi.org/10.1007/s00419-014-0862-x>