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Cable tension estimation by Rayleigh's method considering restraint boundary conditions

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Abstract: This paper introduces a method for accurately estimating cable tension by combining the energy approach with the cable's mode shape. The method simultaneously accounts for the cable's bending rigidity and the rotational stiffness at both ends. Rayleigh's energy-based method is applied to analytically derive a formula for cable tension, while the mode shape is approximated using a nonlinear regression analysis algorithm. The accuracy of the method is validated through comparison with available experimental data. The approach is then applied to the An Dong Extradosed Bridge in Vietnam, demonstrating its effectiveness in evaluating cable forces in similar bridge structures. Notably, a significant difference of up to 13.13% in cable forces is observed when considering the rotational stiffness at the cable ends, highlighting the importance of this factor in structural analysis.

Keywords: Cable tension; Shape function; Rayleigh's method; Rotational stiffness; Nonlinear regression analysis.

1. Introduction

The cable forces must be measured accurately during construction and maintenance stages in the cable system bridges [1, 2]. Cable tension, currently, can be ascertained through direct and/or indirect measurement methods. In term of direct measurement, tension of the cable is measured directly utilizing dedicated load-measurement devices (e.g., hydraulic jacks, pressure and displacement meters), the cable forces extracted from this technique, for example lift-off test, are highly converging (less than 2% difference) to the design tension [3]. The cost of conducting measurement, however, would be prohibitively expensive because of using not only advanced instruments but also skillful labors, and other problems like installed difficulties or poor endurance of sensors are come across in practical applications. To alleviate this, the vibration-based method, one of the indirect measurement methods, is prioritized alternative to practical applications because of its simplicity, speediness and economic efficiency [4]. In vibration-based method, the values of cable tension are commonly calculated through field-measured natural frequencies along with geometrical and mechanical parameters of the cable.

Studies on cable tension by the vibration method have been received the consideration of various authors with disparate approaches, both analytical and numerical analysis to derive cable tension formulas. The sharp distinction of these estimated cable equations, basically, depend on given dynamic cable model and affected cable parameters (e.g., antisymmetric or symmetric in-plane model, cable flexural rigidity, boundary conditions). From that point of view, the existing formulas can be classified straightforwardly into four categories: Taut string theory [5-7]; Considering bending stiffness [8-15]; Cable model with sag extensibility [5, 8]. Cable forces simultaneously accounting for cable sag and bending. stiffness [2, 16-19]. The practical applications of the formulas derived from simplified transversely vibrating string theory, Eq. (1) [6] can trigger huge errors owing to neglecting cable flexural rigidity and sag effect,

$$T = \frac{4ml^2f_n^{\ 2}}{n^2} \tag{1}$$

where f_n, T, m, l and n refer to n_{th} measured natural frequency cable tension, mass per unit length of cable, and the mode number of the cable, respectively. In a case study conducted by Casas [1], the author mentioned that the Eq. (1) can only be applied if the seven lowest frequencies, measured with 0.5% accuracy, lie on a proportional line of natural frequency – mode number diagram. That requirement regarding a total signal length of six-minute is not realistically available for a free-damped vibration of actual cables because free-vibrating time of the cable is limited due to decay of vibration [1]. So far, the vibration-based cable formulas obtained in three final categories have been widely applying to assess cable forces of various cable-stayed bridge in the world.

However, the aforementioned methods have two key limitations. First, they rely on complex frequency equations that require numerous iterations due to eigenproblem constraints. Ren et al. [8], introduced an energy-based tension expression without addressing eigenproblems, but their approach used a fixed-fixed beam mode shape, neglecting axial tension effects and other boundary conditions. Second, previous studies typically assume purely hinged or fixed ends, whereas actual cable behavior depends on anchorage type, support configuration, and anchoring method (Fig.1). Incorrect boundary assumptions can lead to significant force estimation errors, particularly for short or long cables. In practice, elastic supports (rubbers) are often installed at anchorages to reduce bending effects and resist external vibrations, making rotational restraints more realistic boundary conditions. While studies by Ceballos and Prato [9] have considered such conditions, their methods still involve transcendental equations requiring extensive iterations. These limitations complicate practical applications, making accurate cable tension estimation challenging for field engineers when actual boundary conditions deviate from assumptions.

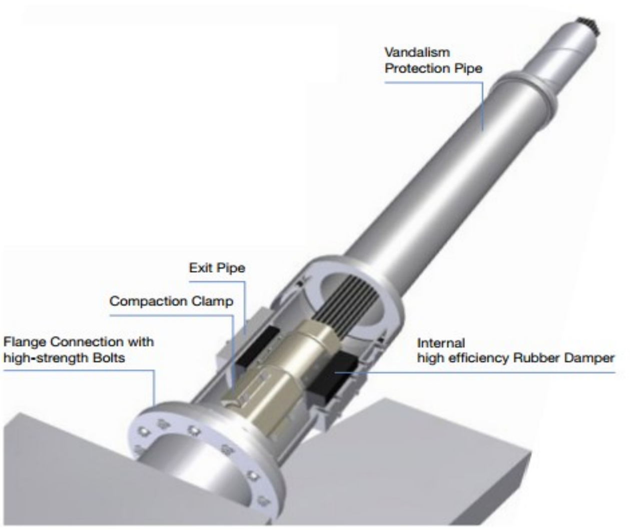


Fig. 1. Elastic support of cable

This paper presents an investigation on energy approach and shape function for estimation of cable tension, in which bending stiffness and various arbitrary boundary conditions, including rotational stiffness at cable ends, are taken into account. The simplified formula of cable force is constructed based on energy principle, so-called Rayleigh's method, without solving eigenproblems of dynamic motion, hence the mathematical complexity made about transcendental equations is significantly reduced. Shape function of the cable, a crucial part of Rayleigh's method, is approximately predicted through an optimization algorithm using a nonlinear regression analysis in accordance with axially loaded beam theory. Also, the schematic diagram and the algorithm of axially loaded beam are presented to analytically derive formulas of rotational stiffness at cable ends. The accuracy of the proposed method is verified through the experiments conducted by Shinke et al. [20]. The calculated results show that the effect of assumed boundary conditions, hinged or fixed ends or rotational restraints on cable tension are extremely remarkable. An application of present method herein is shown on a selected Extradosed bridge, An Dong Bridge, crossing Phan Rang river in Vietnam. It shows that the distinction in term of cable forces between with and without considering rotational stiffness reaches a threshold of 13.13%.

2. Tension formulation by Energy Approach

2.1. Rayleigh's Method

Analytical schematic diagram of in - plane transverse motion of cable with bending stiffness is considered, as shown in Fig. 2. In such case, the cable contains rotationally elastic supports, which are situated nearby the anchorages, assuming that free length of cable, l_1, and deviation angle are marked from the deviators; therefore, the length of the elastic supports l_0 at points A and B is neglected in the analysis. These elastic supports at A and B are treated as elastic springs with constant stiffness K_A and K_B, respectively. Without the elastic supports, cable model turns back conventional analysis under hinged or fixed boundaries ($K_A = K_B = 0$).

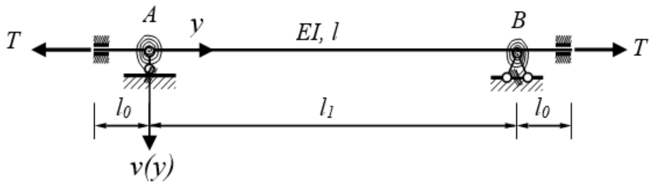


Fig. 2. Equivalent model of cable with rotational restraints

The concept in the Rayleigh's method is based on the principle of conservation of energy. The frequency of vibration can be found by equating the maximum potential energy developed during the motion to the maximum kinetic energy.

The potential energy of this system, Eq. (2), include entirely the energy of the axial load, flexural deformation of the cable and strain energy of two springs.

$$V = \frac{1}{2}\int_0^{l_1} EI[\frac{\partial^2 v(y,t)}{\partial y^2}]^2 dy + \frac{T}{2}\int_0^{l_1}[\frac{\partial v(y,t)}{\partial y}]^2 dy + \frac{1}{2}K_A\left[\left.\frac{\partial v(y,t)}{\partial y}\right|_{y=0}\right]^2 + \frac{1}{2}K_B\left[\left.\frac{\partial v(y,t)}{\partial y}\right|_{y=l_1}\right]^2 \quad (2)$$

Kinetic energy of distributed mass is given by

$$T = \frac{1}{2}\int_0^{l_1} m[\frac{\partial v(y,t)}{\partial t}]^2 dy \quad (3)$$

The displacement of the generalized coordinate in free vibration of cable in the y-direction is written as the subsequent equation:

$$v(y,t) = \psi(y) \times Z_0 \sin(\omega t) \quad (4)$$

in which $\psi(y)$ is the shape function of cable, which represents the ratio of the displacement at any point y to the reference displacement or generalized coordinate. Equating the maximum potential energy to the maximum kinetic energy gives

$$f^2 = \frac{EI\int_0^{l_1}\left[\frac{d^2\psi(y)}{dy^2}\right]^2 dy + T\int_0^{l_1}\left[\frac{d\psi(y)}{dy}\right]^2 dy + K_A\left[\left.\frac{d\psi(y)}{dy}\right|_{y=0}\right]^2 + K_B\left[\left.\frac{d\psi(y)}{dy}\right|_{y=l_1}\right]^2}{4\pi^2\left(\frac{w}{g}\right)\int_0^{l_1}\psi(y)^2 dy} \quad (5)$$

where K_A, K_B are rotational stiffness at A and B, respectively; and EI, l, T, w, g, f = ω/2π, are the bending stiffness, length, cable tension, weight per unit length, gravitational acceleration, measured fundamental frequency of the cable, correspondingly. Cable tension can be deduced using Eq. (5).

Next sections shall introduce a method to approximate mode shape of the cable, $\psi(y)$, through optimization algorithm, as well as propose formulas of rotational stiffness, K_A and K_B.

2.2. Rotational Stiffness at Cable Ends

This section sets out to introduce proper schemes to derive formulas of static rotational stiffness at one end or both ends of the cable through cable static analysis. The vibration model of the cable with rotationally elastic boundaries symmetrically imposed at both ends is shown in Fig. 2, in which deviators are located at a distance l_0 of lower and upper anchorage, and elastic supports are placed between deviator and exit pipe of the cable. Assumptions that stiffness of elastic support $K_A = K_B = K$ at both cable ends; free length, l_1, and deviation angle of the cable are started from the location of elastic support. For those assumptions, the system become symmetric, and hence, it can be reduced to half structure as indicated in Fig. 3.

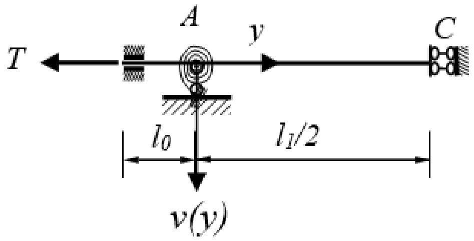


Fig. 3. Half taut-string cable model with a spring

Differential equation:

$$\frac{d^4v(y)}{dy^4} - \frac{T}{EI}\frac{d^2v(y)}{dy^2} = 0 \quad (6)$$

Solution of differential equation

$$v(y) = C_1 + C_2 y + C_3 e^{\alpha y} + C_4 e^{-\alpha y} \quad (7)$$

$$\text{with } \alpha^2 = \frac{T}{EI} \quad (8)$$

where C_1, C_2, C_3 and C_4 are constants. With the purpose of obtaining rotational stiffness at ends of cable, the cable is assumed to subjected to unit moment at deviators. Rotational stiffness is then found by dividing unit moment to rotational angle of cross section at deviator. The constants of Eq. (7) are obtained by substituting v(y) from Eq. (7) into the following boundary conditions, both displacement and force conditions.

$v_{(y=0)} = 0:$

$$C_1 + C_3 + C_4 = 0 \quad (9)$$

$-EIv''_{(y=0)} = 1:$

$$-EI\alpha^2(C_3 + C_4) = 1 \quad (10)$$

$v'_{(y=l_1/2)} = 0:$

$$C_2 + C_3\alpha e^{\frac{1}{2}\alpha l_1} - C_4\alpha e^{-\frac{1}{2}\alpha l_1} \quad (11)$$

$-EIv'''_{(y=l_1/2)} = 0:$

$$-EI\left(C_3\alpha^3 e^{\frac{1}{2}\alpha l_1} - C_4\alpha^3 e^{-\frac{1}{2}\alpha l_1}\right) = 0 \quad (12)$$

Finally, the rotational stiffness at both ends of cable is given as

$$K = \frac{-EIv''_{(y=0)}}{v'_{(y=0)}} = \frac{EI\alpha\left(e^{\frac{1}{2}\alpha l_1} + e^{-\frac{1}{2}\alpha l_1}\right)}{e^{\frac{1}{2}\alpha l_1} - e^{-\frac{1}{2}\alpha l_1}} \quad (13)$$

where α refers to Eq. (8); and EI and l_1 are bending stiffness and free length of cable, respectively. It is noted that field measurements such as vibration tests can be conducted to assess cable behavior. By analyzing the relationship between applied loads and rotational deformations at the supports, the effective rotational stiffness can also be determined.

For cable with single rotational restraint acting at only one end of the cable, the analyzed procedure to explicitly obtain expression of one-end rotational stiffness, K_A, are the same as presented in the first case. The boundary constraints are presented as follows

$v_{(y=0)} = 0:$

$$C_1 + C_3 + C_4 = 0 \quad (14)$$

$-EIv''_{(y=0)} = 1:$

$$-EI\alpha^2(C_3 + C_4) = 1 \quad (15)$$

$v_{(y=l_1)} = 0:$

$$C_1 + C_2 l_1 + C_3 e^{\alpha l_1} + C_4 e^{-\alpha l_1} = 0 \quad (16)$$

$-EIv''_{(y=l_1)} = 0:$

$$-EI\left(C_3\alpha^2 e^{\alpha l_1} + C_4\alpha^2 e^{-\alpha l_1}\right) = 0 \quad (17)$$

the rotational stiffness at one end of cable is obtained as

$$K_A = \frac{-EIv''_{(y=0)}}{v'_{(y=0)}} = \frac{EI\alpha^2 l_1\left(e^{\alpha l_1} - e^{-\alpha l_1}\right)}{-e^{\alpha l_1} + e^{-\alpha l_1} + \alpha l_1 e^{-\alpha l_1} + \alpha l_1 e^{\alpha l_1}} \quad (18)$$

2.3. Shape Function of Cable

Shape function of cable is a function that describe configurations of cable as accurate as possible when cable deform. Assumed shape function, firstly, must satisfy all possible displacement boundary conditions. In fact, there are a large number of assumed shape functions meet geometric boundary conditions of the system; therefore, in some simple cases, shape function can be assumed directly, on the basis of practical experiences of analyzers, without any intensive analysis. Consequently, if mode shape of cable is predicted inappropriately, it can bring about unaccepted errors in prediction of cable tension from Rayleigh's Method. In other words, a shape function that satisfies only the geometric boundary conditions does not always ensure an accurate result for the estimation of cable tension. For that important reason, the method introduced herein allow estimating mode shape of cable, from static analysis of cable, not only satisfying displacement boundary condition but also covering force conditions and static differential equation of the cable. Practically, two typical shape functions often used are polynomial and trigonometric functions. In this paper, the former which is demonstrated as Eq. (19) has been selected due to its simplicity, computational efficiency, and ability to approximate cable deformation with sufficient accuracy.

$$\psi(y) = a_0 + a_1 y + a_2 y^2 + ... + a_i y^i + ... + a_n y^n \quad (19)$$

in which a_1, a_2, …, a_n are constants of the shape function. The algorithm to ascertain these constants is given in next section.

2.4. Optimization of Shape Function

This section puts forward optimization algorithm, so-called nonlinear regression analysis to reasonably identify the constants of shape function, Eq. (19). As mentioned earlier, the accuracy of cable tension estimated using Rayleigh's method depends heavily on the shape function of the cable. Addressing the question of how a reasonable shape function can be selected to ensure good results, becoming the most interested. The common approach is that an appropriate mode shape can be determined as the deflected shape due to selected set of static forces. Two general static forces are inertia forces at each time instant or self-weight of the cable applied to the cable in an appropriate direction. The former is not helpful because inertia forces involve with unknow shape. As a result, the proposed algorithm herein shall predict mode shape of the cable, Eq. (19), as deflected shape of cable subjected to its self-weight, as well as simultaneously satisfy geometric boundary conditions, force boundary conditions and differential equation of cable.

The differential equation representing the static profile of the cable, accounting for bending stiffness, under its self-weight is shown as

$$\frac{d^4\psi(y)}{dy^4} - \frac{T}{EI}\frac{d^2\psi(y)}{dy^2} = -\frac{q(y)}{EI} \quad (20)$$

where q(y) refer to uniformly transverse load.

Mode shape of the cable is obtained after normalizing displacement of cable to reference displacement. The nonlinear regression analysis is used to properly identify coefficients (a_i) of the mode shape. The content of regression analysis, carried out in this study, is that with any value of y in the domain of shape function, the parameters (a_i) from Eq. (19) are ascertained to minimize the sum of squared error (SSE) between data values and predicted values. The procedures are following three steps:

Step 1. Set a target function: Target function is designed to minimize the sum of squared error (SSE) in coincide with nonlinear regression algorithm.

$$SSE = \sum_{i=1}^{n}\left(f_{pi} - f_i\right)^2 \to Min \quad (21)$$

in which f_{pi} is the values predicted by regression model; and f_i refers to data values. Knowing differential equation of the cable, Eq. (20), predicted values of f_{pi} and data values of f_i can be deduced as sequent expressions. It is recommended that the objective function values range from 10^{-6} to 10^{-4} indicating good convergence of the analysis.

$$f_{pi} = \left(\left. \frac{d^4\psi(y)}{dy^4} \right|_{y=y_i} - \frac{T}{EI} \left. \frac{d^2\psi(y)}{dy^2} \right|_{y=y_i} \right) \quad (22)$$

$$f_i = \left. \frac{q(y)}{EI} \right|_{y=y_i} \quad (23)$$

Step 2. Set constraints for target function: The constraints of target function consist of geometric and force boundary conditions of the cable model. Three cases of boundary conditions, including hinged, fixed and rotational restraint ends, are investigated correspondingly.

Case 1: Hinged end conditions

$$\psi(0) = 0; \psi(l) = 0 \quad (24a)$$

$$EI \left. \frac{d\psi^2(y)}{dy^2} \right|_{y=0} = 0; EI \left. \frac{d\psi^2(y)}{dy^2} \right|_{y=l} = 0 \quad (24b)$$

Case 2: Fixed end conditions

$$\psi(0) = 0; \psi(l) = 0 \quad (25a)$$

$$\left. \frac{d\psi(y)}{dy} \right|_{y=0} = 0; \left. \frac{d\psi(y)}{dy} \right|_{y=l} = 0 \quad (25b)$$

Case 3: Rotational restraint end conditions

$$\psi(0) = 0; \psi(l_1) = 0 \quad (26a)$$

$$K_A \left. \frac{d\psi(y)}{dy} \right|_{y=0} = EI \left. \frac{d\psi^2(y)}{dy^2} \right|_{y=0}; \quad (26b)$$

$$K_B \left. \frac{d\psi(y)}{dy} \right|_{y=l_1} = -EI \left. \frac{d\psi^2(y)}{dy^2} \right|_{y=l_1}; \quad (26c)$$

where K_A, K_B are elastic stiffness of rotational restraints at the corresponding cable ends. The value of these stiffness is computed using Eq. (13) or Eq. (18). The degree of fixity in the supports can also be represented through the non-dimensional parameters k_a and k_b [9].

$$k_a = \frac{K_A l}{K_A l + \pi^4 EI}; k_b = \frac{K_B l}{K_B l + \pi^4 EI} \quad (27)$$

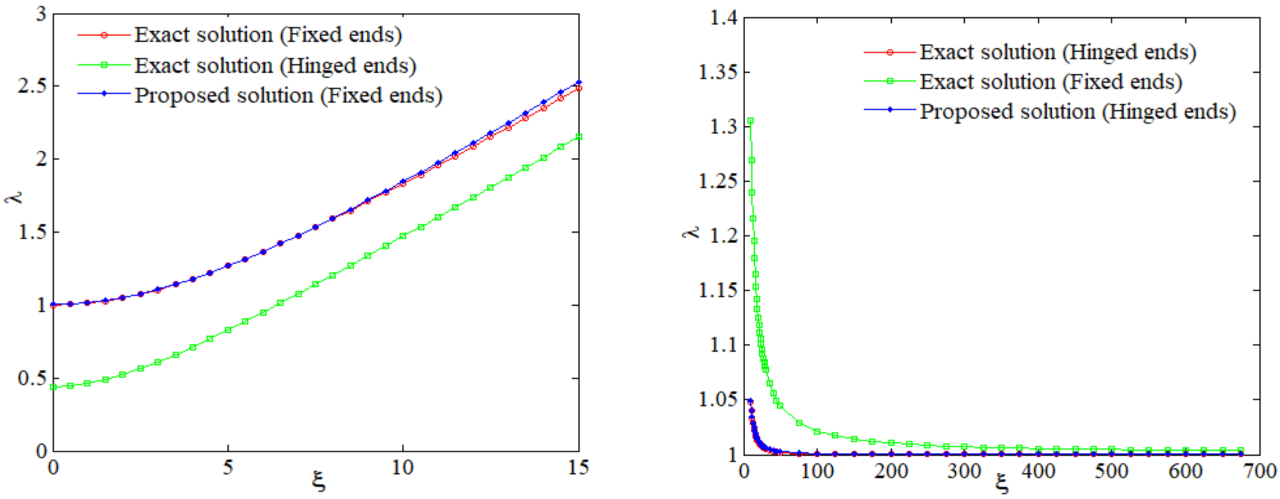
Step 3. Iteration: Entering initial guessed values of a_i of shape function for regression model coefficients, then run the iterative steps. In this paper, the arbitrary values of 1 for each coefficient are used. Better initial guessed values will speed up iterative process, but the arbitrary guesses provide a good test of the ability of iteration – supported tools. The iterative process shall be stopped after a huge number of consecutive loops. As a result, the reasonable coefficients of shape function of the cable shall be found if both the target function and all boundary conditions are satisfied. Next section will shortly introduce a powerful iteration tool called Excel Solver to help conducting speedily and conveniently the optimization of cable shape function.

3. Numerical Results and Discussions

In this section, firstly, cable tension was estimated utilizing Eq. (5) for three cases: the cable model with hinged end conditions, fixed ones and rotational restraints at both ends. In which, mode shape of the cable is predicted in the form of Eq. (19). Secondly, the feasibility of the proposed method is confirmed by comparing with exact solution. Then, the accuracy of the proposed method in estimation of cable forces is verified by comparison with available experiment results. Finally, an investigation on rotational restraints for evaluation of cable tension were presented. Explanations of numerical results and discussion are following points:

3.1. Compare to the exact solution

A dimensionless parameter $\xi = l(T/EI)^{1/2}$ is introduced to investigate the simultaneous influence of bending stiffness and cable tension on the vibration behavior of the cable. Fig. 4(a) and 4(b) present the ratio of fundamental frequency of proposed method and exact solution to based frequency of beam (small ξ) and taut string theory (large ξ), respectively. In Fig.4(a), $\lambda = f_p/f_b$ whereas $\lambda = f_p/f_{st}$ in Fig.(4b). The detailed expressions of these parameters refer to Table 1.



(a) (b)

Fig. 4. Ratio of fundamental frequency versus ξ: (a) small ξ; (b) large ξ

Table 1. Expressions and features used in investigation

	Small ξ	Large ξ
Boundary condition	Fixed ends	Hinged ends
Fundamental frequency using proposed method (f_p)	Refer to Eq. (5)	Refer to Eq. (5)
Fundamental frequency using exact solution (f_p)	$2(\alpha l)(\beta l)[1-\cos(\alpha l)\cosh(\beta l)]$ $+\ [(\beta l)^2-(\alpha l)^2]\sin(\alpha l)\sinh(\beta l)$ where $\alpha^2=(\zeta^4+\gamma^4)^{1/2}-\zeta^2;\beta^2=(\zeta^4+\gamma^4)^{1/2}+\zeta^2$ $\zeta^2=T/(2EI);\ \ \gamma^4=4\pi^2 f_p^{\,2} w/(gEI)$	$f_p=\sqrt{(T+\frac{\pi^2 EI}{l^2})\frac{g}{4wl^2}}$
Fundamental frequency of beam theory (f_b)	$f_b=\frac{\pi}{2}\sqrt{\frac{gEI}{wl^4}}$	No use
Fundamental frequency of taut string theory (f_b)	No use	$f_{st}=\frac{1}{2}\sqrt{\frac{gT}{wl^2}}$

For small ξ, which corresponds to short cables, the natural frequency tends to align with the values predicted by beam theory under fixed-end conditions. As shown in Fig. 4(a), fixed-end boundary conditions provide a more accurate estimation of cable tension for short cables compared to hinged-end conditions. Using hinged condition can cause unaccepted overestimation of cable forces. Proposed method highly coincides with exact solution, error around 1%, for the region $0 < \xi < 10$.

In case of large ξ as shown in Fig. 4(b), regarding long cable, the dynamic characteristics of cable close to a taut string theory. For $\xi > 100$, the natural frequencies of the cable are nearly same as its of string theory, with the difference of 0.1%. Also, hinged end conditions seem to be more realistic rather than fixed ends when predict cable tension through vibration analysis of long cable. Both exposed formula and exact solution agree extremely well to each other within the error of less than threshold of 0.1%.

One particularly noticeable point from Fig. 4 is that even small or large value of ξ, the discrepancy in dynamic behavior of cable between hinged and fixed end boundary conditions are significant, especially short cable.

3.2. Compare to the experiment

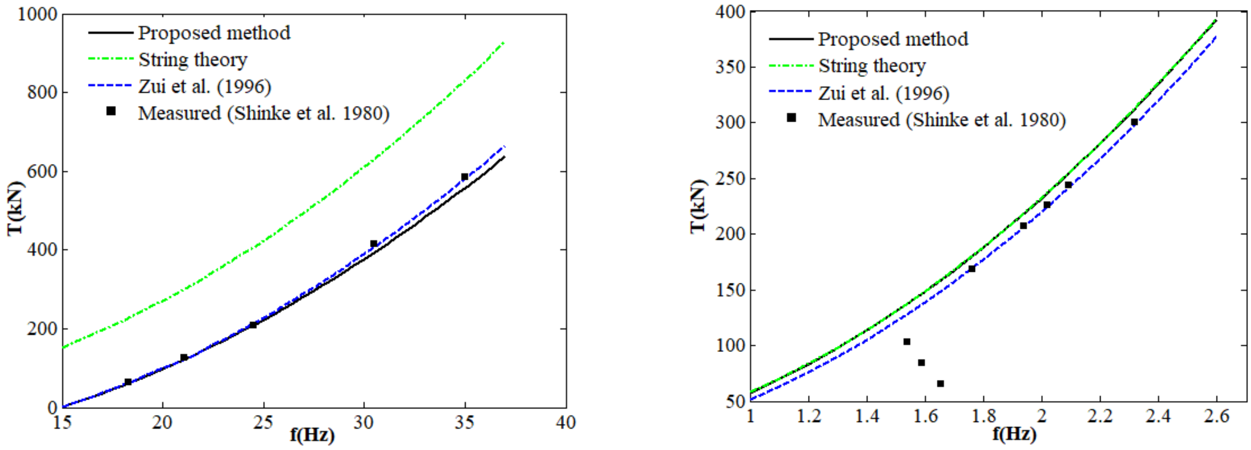
The experiments on cable tension were conducted by Shinke et al [20]. The short cable of

3.4m and long cable of 31.5m from experiment were chosen for validation of feasibility of proposed method. The properties of the experimental cables as shown in Table 2.

Table 2. Cable properties

l (m)	w (kN/m)	EI (kN.m^2)
3.4	0.144	34.5
31.5	0.144	34.5

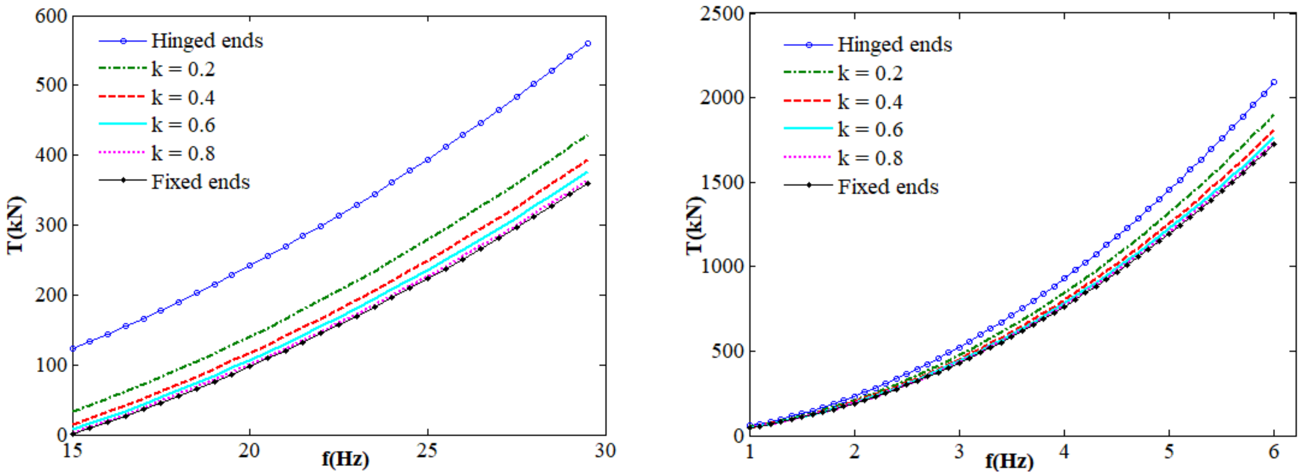
Cable tension employing present method are in a good agreement with those from the experiment with the error of around 4% (Fig.5). It also matched well on a wide range of frequencies with those using formula proposed by Zui et al [2]. One point to consider when estimation of cable tension is that cable forces calculated from taut string theory beget extreme error in case of short cable (Fig.5a). By comparing estimated cable forces between proposed method and available experimental results in both short and long cable, the accuracy and feasibility of present method are validated.



(a) (b)

Fig. 5. Tension forces: (a) 3.4 m-long cable; (b) 31.5 m-long cable

3.3. Effect of the rotational on cable tension



(a) (b)

Fig. 6. Impact of boundary conditions on cable tension: (a) small ξ; (b) large ξ

The cable tension accounting for bending stiffness and rotational restraint ends are investigated utilizing present method as shown in Fig. 6. In this investigation, the non-dimensional parameters of rational stiffness k_a and k_b shown in Eq. (27) are used. The values of k_a and k_b take from 0 to less than 1. If $k_a = k_b = 0$, boundary conditions are hinged ends. The degree of rotational restraints shall increase with an increase in the value of k_a and k_b. Boundary condition is treated as fixed ends if k_a and k_b close to 1. Figs 6(a) and 6(b) shows effect of assumed boundary conditions on cable tension of 3.4m - long cable 31.5m – long cable. When the natural frequency of cable is high enough, the gaps of cable tension between fixed ends, hinged ends and rotational end restraint are enormously lager. For example, when ξ is around 10, corresponding to approximated frequency of 28 Hz, the discrepancy in term of cable forces between hinged ends and fixed ends are surrounding 38%.

3.4. Case study: An Dong-Extradosed Bridge

An Dong Bridge is crossing Dinh river of Ninh Thuan province, Vietnam. It connects Ninh Hai commune, Ninh Phuoc, to Dong Hai ward, Phan Rang – Thap Cham, and carry four traffic lanes. The total length of An Dong Bridge is 1018 meters, in which main bridge are an extradosed structure with the length of 580 meters. The approach spans of An Dong Bridge are simple beam structure.

An Dong extradosed Bridge has a three cell box girders cross-section that is supported in five spans by extradosed cable in a twin cable of fan – harp shaped layout with 48 pairs of stay cables, Fig. 7

The cable tensions of some typical cables of An Dong extradosed Bridge were evaluated, using formula of proposed method herein and practical formula introduced by Zui et al [2]. In term of cable anchorage to the box-girder and towers, according to configuration of the anchorage at girder, Fig. 8, rubbers were placed at the deviators. The cables were also not terminated at the towers. It passed through the tower over the saddle. This such anchorage type at girder and anchoring method to the towers led to boundary condition of cable becoming imperfect hinged or fixed condition. Therefore, the rotational restraints at cable ends were considered in force determination of selected cables in order to increase the accuracy of calculated forces. For more simplicity, In this cable tension evaluation, the vibration cable model was assumed with rotationally elastic boundaries symmetrically imposed at both ends.

Six typical extradosed cable as indicated in Fig.7, namely C-01, C-02, C-03, C-04, C-05 and C-06, were selected for cable force estimation. The geometrical and mechanical properties of these selected cables are indicated in Table 3. It shows that the rotational stiffness calculated applying proposed formula and formula presented by Ceballos and Prato [9] were in good agreement with the error of about 1.32%.

To record the dynamic response of the extradosed cables, a three – dimensional acceleration transducer, namely, ARF-20A which was supplied by Tokyo Sokki Kenkyujo Co.,Ltd, was employed. Frequency range of the transducer reach to maximum of 80 Hz, with sensitivity deviation of 5%. The recorded signals, then, were converted and processed through a dynamic measurement equipment, SDA-7910. To capture different frequencies of the cable, an algorithm called Fast-Fourier-Transform (FFT) is used with sampling frequency of 50 Hz. FFT is an efficient implementation of the Discrete-Fourier-Transform (DFT), converting cable response from time domain representation into frequency domain representation. Based on that frequency domain, different natural frequencies of the cable were extracted. Hanning window function were used to reduce spectral leakage of the signals. Consequently, power spectrums of extradosed cables, from C-01 to C-06, are shown in Fig. 9, respectively, in which the value of fundamental frequency is marked at the first peak of the corresponding dominant spectrums. Cable tension of selected cables are indicated in Table 4.

Table 4 illustrates cable forces of six selected extradosed cable of An Dong Bridge using formula of proposed method herein and practical formula introduced by Zui et al. [2]. The former accounted for rotational restraints at both ends of the cables, whereas the latter assumed boundary conditions as fixed ends. Consequently, the different assumed boundary conditions between two methods posed a discrepancy in the results of cable forces, a maximum of 13.13% among these selected cables. It is noted that the rotational restraints provide additional stiffness to the cable system. In the Zui et al. [2] model, the natural frequency of the cable is influenced by both cable tension and bending stiffness, whereas in the proposed model, rotational stiffness also contributes to the natural frequency. At the same measured natural frequency, the cable forces estimated using the proposed method are consistently lower than those obtained using the Zui et al. [2] model.

Fig. 7. An Dong extradosed Bridge: Source from Sbtech Joint Stock Company, Vietnam

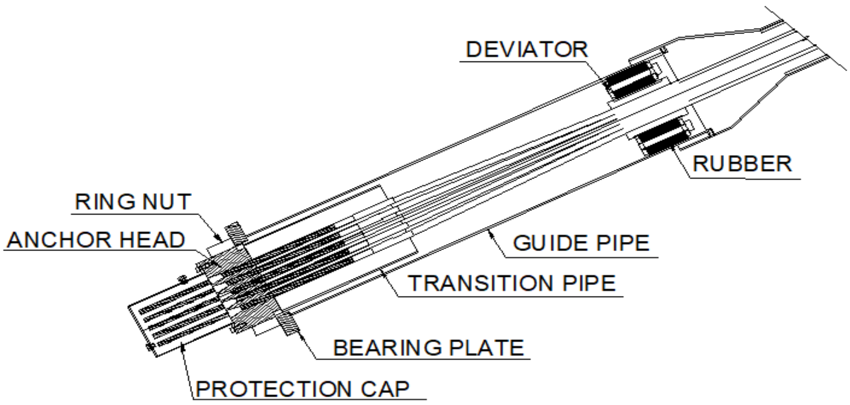
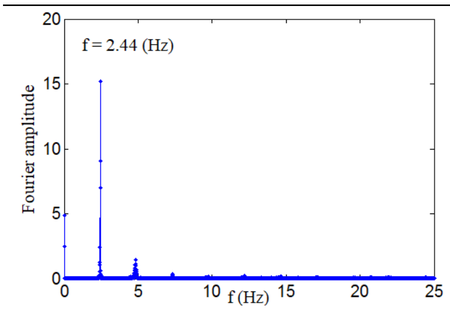


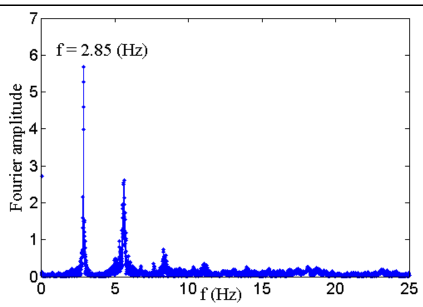
Fig. 8. Stay anchorage configuration with elastic support, rubbers, of An Dong extradosed Bridge

Table 3. Geometrical and mechanical properties of extradosed cables of An Dong Bridge

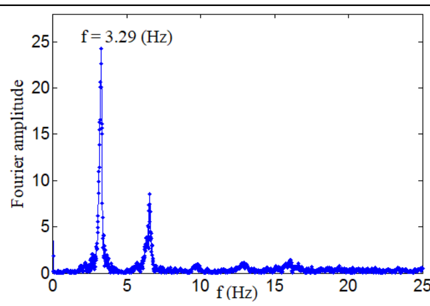
Cable	strands (n)	length l_1 (m)	w (kN/m)	EI (kN.m^2)	Proposed formula K (kN.m/rad)	Ceballos and Prato [9] K (kN.m/rad)
C-01	37	55.50	0.47	514.75	1236.86	1246.21
C-02	37	47.90	0.47	514.75	1248.67	1259.51
C-03	37	40.60	0.47	514.75	1224.11	1236.92
C-04	37	33.20	0.47	514.75	1220.07	1235.78
C-05	37	26.00	0.47	514.75	1174.09	1194.23
C-06	37	19.70	0.47	514.75	1128.72	1155.47



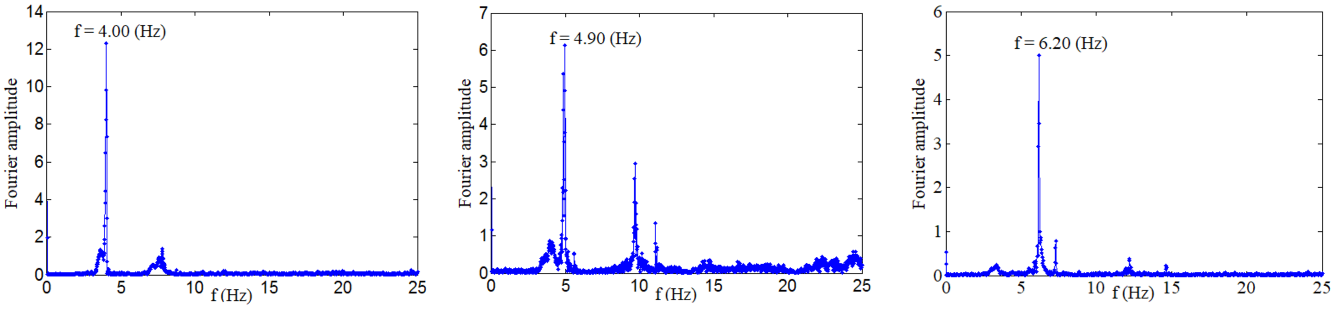
(a) C-01



(b) C-02



(c) C-03



(d) C-04 (e) C-05 (f) C-06

Fig. 9. Power spectrums of response of six selected cables

Table 4. Cable forces of six selected cables

Cable	Natural frequency f (Hz)	Proposed method T (kN)	Zui et al [2] T (kN)	Difference (%)
C-01	2.44	2972.10	3421.13	13.13
C-02	2.85	3029.21	3460.34	12.46
C-03	3.29	2911.15	3288.05	10.11
C-04	4.00	2891.86	3216.95	13.13
C-05	4.90	2678.09	2906.56	7.86
C-06	6.20	2475.23	2595.86	4.65

4. Conclusions

This paper presents a method for estimating cable tension using an energy-based approach that accounts for cable bending stiffness and rotational restraints at the cable ends. An optimization algorithm employing nonlinear regression analysis was introduced to accurately approximate the cable mode shape, and practical formulas for rotational stiffness were derived. The estimated cable forces obtained using the proposed method show strong agreement with both exact solutions and experimental data. The influence of boundary conditions is particularly significant for short cables, where variations in cable force are more pronounced compared to longer cables. When the parameter ξ is small, the estimated tension approaches the values of a fixed-fixed end cable, whereas for sufficiently large ξ, the forces align with those of a hinged-hinged end condition. At high natural frequencies, the discrepancy in cable tension between fixed ends, hinged ends, and rotational end restraints becomes excessively large. The effectiveness of the proposed method was demonstrated in the evaluation of cable forces on the An Dong Extradosed Bridge in Vietnam, where the difference in cable tension with and without considering rotational stiffness reached a threshold of 13.13%.

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