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Effects of lattice stiffeners on the nonlinear behavior of variable-curvature FG-GRC panels subjected to blast and step loads

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Abstract: This paper presents a semi-analytical investigation of the nonlinear dynamic responses of lattice-stiffened variable-curvature functionally graded graphene-reinforced composite (FG-GRC) sandwich panels with a porous core under impulsive loading conditions. The structure consists of FG-GRC face sheets and a lightweight porous core, with material properties varying continuously through the thickness according to predefined distribution patterns. A refined smeared stiffener approach is developed to incorporate the mechanical contributions of different lattice configurations, accounting for higher-order shear deformation effects. The theoretical formulation is established based on the higher-order shear deformation theory (HSDT) combined with von Kármán geometric nonlinearity. To address the complexity of the variable-curvature geometry, an approximate stress function satisfying the compatibility condition is introduced within a Galerkin framework. The resulting nonlinear governing equations are derived using the Euler-Lagrange equations and solved via a fourth-order Runge-Kutta scheme to obtain time-dependent responses. Numerical results are presented to evaluate the influences of lattice stiffener parameters, core porosity distribution, graphene gradation, curvature profiles, and loading types (blast and step loads) on the dynamic behavior. The findings reveal that the integration of lattice stiffeners significantly enhances the structural stiffness and mitigates deflection amplitudes, while the presence of a porous core introduces a trade-off between weight reduction and structural rigidity. The interaction among stiffener topology, material gradation, and geometric curvature is shown to play a critical role in the dynamic performance of the proposed structures.

Keywords: Lattice stiffener, Functionally graded graphene reinforced composite, Impulsive load, Higher-order shear deformation theory, Variable-curvature panel.

1. Introduction

Structural components such as rectangular and circular plates, cylindrical panels, and

spherical shells are extensively employed in advanced engineering applications. As a result, their mechanical performance has been widely

investigated, with particular emphasis on static stability, dynamic behavior, and vibration characteristics.

To address these problems, a variety of theoretical models and solution techniques have been developed. In the context of functionally graded materials (FGMs) and sandwich structures, numerous studies have examined linear and nonlinear vibrations, postbuckling responses, dynamic instability, and static deformation of plates and shell-type structures [1-7]. These analyses have been conducted using different theoretical frameworks, including classical elasticity theory, Donnell shell theory, first-order shear deformation theory (FSDT), higher-order shear deformation theory (HSDT), and refined shear deformation models. Various analytical and numerical methods, such as the Ritz method, Chebyshev-Ritz method, Galerkin approach, perturbation techniques, Fourier series expansions, and differential quadrature methods (DQM), have also been employed. In addition, the mechanical and thermal buckling behaviors of stiffened plates and porous-core FGM structures, including circular plates and spherical shells, have been investigated using approaches based on FSDT, HSDT, and improved smeared stiffener techniques [8-10].

With the emergence of advanced nanocomposites, functionally graded carbon nanotube-reinforced composites (FG-CNTRCs) have attracted significant attention due to their superior mechanical properties. Extensive research efforts have been devoted to understanding their structural behavior under various loading conditions. In particular, large-amplitude vibration responses of CNT-reinforced plates [11, 12] and cylindrical panels [13] have been investigated using perturbation-based approaches. Advanced meshless techniques, such as the kp-Ritz and improved moving least-squares Ritz (IMLS-Ritz) methods, have been widely applied to examine free vibration [14], buckling [15], and postbuckling behaviors [16] of CNT-reinforced plates. Furthermore, natural frequency

characteristics of CNT-reinforced cylindrical [17, 18] and conical panels [19] have been systematically analyzed. The isogeometric approach has also been employed to study the nonlinear flutter behavior of FG-CNTRC quadrilateral plates subjected to supersonic airflow [20]. In addition, active vibration control strategies for CNT-reinforced plates integrated with piezoelectric layers have been developed using advanced element-free formulations based on IMLS-Ritz techniques within the framework of HSDT [21, 22].

Owing to its two-dimensional atomic structure, graphene exhibits superior interfacial bonding with the surrounding matrix compared to carbon nanotubes, leading to enhanced load transfer efficiency in composite materials. As a result, functionally graded graphene-reinforced composites (FG-GRCs) have emerged as a promising class of advanced materials for next-generation engineering applications. Numerous studies have been conducted to evaluate the mechanical performance of FG-GRC structures. In particular, bending, buckling, and postbuckling behaviors of FG-GRC plates have been extensively analyzed using HSDT and perturbation-based approaches [23, 24]. The vibration and nonlinear vibration characteristics of such structures, including cylindrical panels, have also been investigated within similar theoretical frameworks [25-27]. In addition, isogeometric finite element methods have been employed to examine thermal buckling, postbuckling, and free vibration responses of FG-GRC plates [28-30]. More recently, the incorporation of stiffeners into FG-GRC plates and cylindrical panels has been explored, with improved smeared stiffener techniques developed to analyze their stability and postbuckling behavior [31-34]. Nam et al. [35] studied the nonlinear static and dynamic buckling behavior of variable-curvature composite panels reinforced graphene platelets with porous core.

Although significant progress has been made in the analysis of FG-GRC structures, existing

studies are mainly restricted to unstiffened or conventionally stiffened configurations without considering porous-core sandwich systems. Moreover, the nonlinear dynamic behavior of such structures under impulsive loads has received limited attention. In particular, the combined effects of lattice stiffeners, material gradation, porous core distribution, and variable curvature have not been systematically investigated.

To address these limitations, the present study develops a semi-analytical framework for analyzing the nonlinear dynamic responses of lattice-stiffened variable-curvature FG-GRC sandwich panels with a porous core subjected to blast and step loads. An improved smeared stiffener technique is proposed to account for the mechanical contributions of different lattice configurations, while the material properties of the FG-GRC layers and porous core are modeled through appropriate distribution functions. The formulation is established based on the higher-order shear deformation theory (HSDT) incorporating von Kármán geometric nonlinearity. By employing a Galerkin approach in conjunction with the Lagrange principle, the governing equations of motion are derived and subsequently solved using a fourth-order Runge–Kutta scheme. The effects of key parameters, including lattice stiffener characteristics, graphene distribution patterns, core porosity, curvature profiles, and loading types, are systematically examined to provide new insights into the design and optimization of advanced composite structures subjected to extreme dynamic environments.

2. Structure and material designs

The orientation of graphene is defined along the longitudinal direction of the panel, corresponding to the 0° layer, while the circumferential direction is associated with the 90° layer. Both the face sheets and the stiffeners are constructed from multiple graphene-reinforced composite (GRC) layers. Three different graphene distribution patterns are examined, namely uniform distribution (UD), FG-X, and FG-O. For the

functionally graded cases, the graphene volume fraction varies symmetrically with respect to the mid-plane of the panel. Specifically, the FG-O distribution follows the sequence $[0.11/0.09/0.07/0.05/0.03]_s$, while the FG-X distribution is defined as $[0.03/0.05/0.07/0.09/0.11]_s$. In contrast, for the UD configuration, the graphene volume fraction remains constant across all layers ($V_G = 0.07$).

The shallow FG-GRC panels are considered in three different geometrical configurations, namely cylindrical, sinusoidal, and parabolic forms, and are described within a quasi-Cartesian coordinate system, as illustrated in Fig. 1. The geometric characteristics of the panels are defined by the total thickness h , the central rise χ , the longitudinal dimension a , and the circumferential dimension b . In this study, the core layer is assumed to be composed of the same material as the matrix phase. The effective properties of the porous core, including the Young's modulus, coefficient of thermal expansion, and Poisson's ratio, are evaluated based on appropriate homogenization relations as follows [35]

$$\begin{cases} E_{pc} = E_M \left[1 - e_0 \cos\left(\frac{\pi z}{h_{pc}}\right) \right], -\frac{h_{pc}}{2} < z < \frac{h_{pc}}{2}, \\ \alpha_{pc} = \alpha_M \\ \rho_{pc} = \rho_M \left[1 - (1 - \sqrt{1 - e_0}) \cos\left(\frac{\pi z}{h_{pc}}\right) \right] \end{cases} \quad (1)$$

where the porosity coefficient is denoted by e_0 ($0 \leq e_0 \leq 1$).

The cosine function is adopted to describe the porosity distribution as it provides a smooth and symmetric variation through the thickness, with lower porosity near the surfaces and higher porosity toward the mid-plane, which is consistent with commonly observed material configurations. For the GRC face sheets, three graphene distribution patterns are considered, namely UD, FG-X, and FG-O. The overall structural system is

composed of GRC face layers, a porous core layer, and GRC stiffeners.

Based on the adopted distribution types,

three corresponding configurations of stiffened panels are defined as UD-PC-UD/UD, X-PC-X/X, and O-PC-O/O.

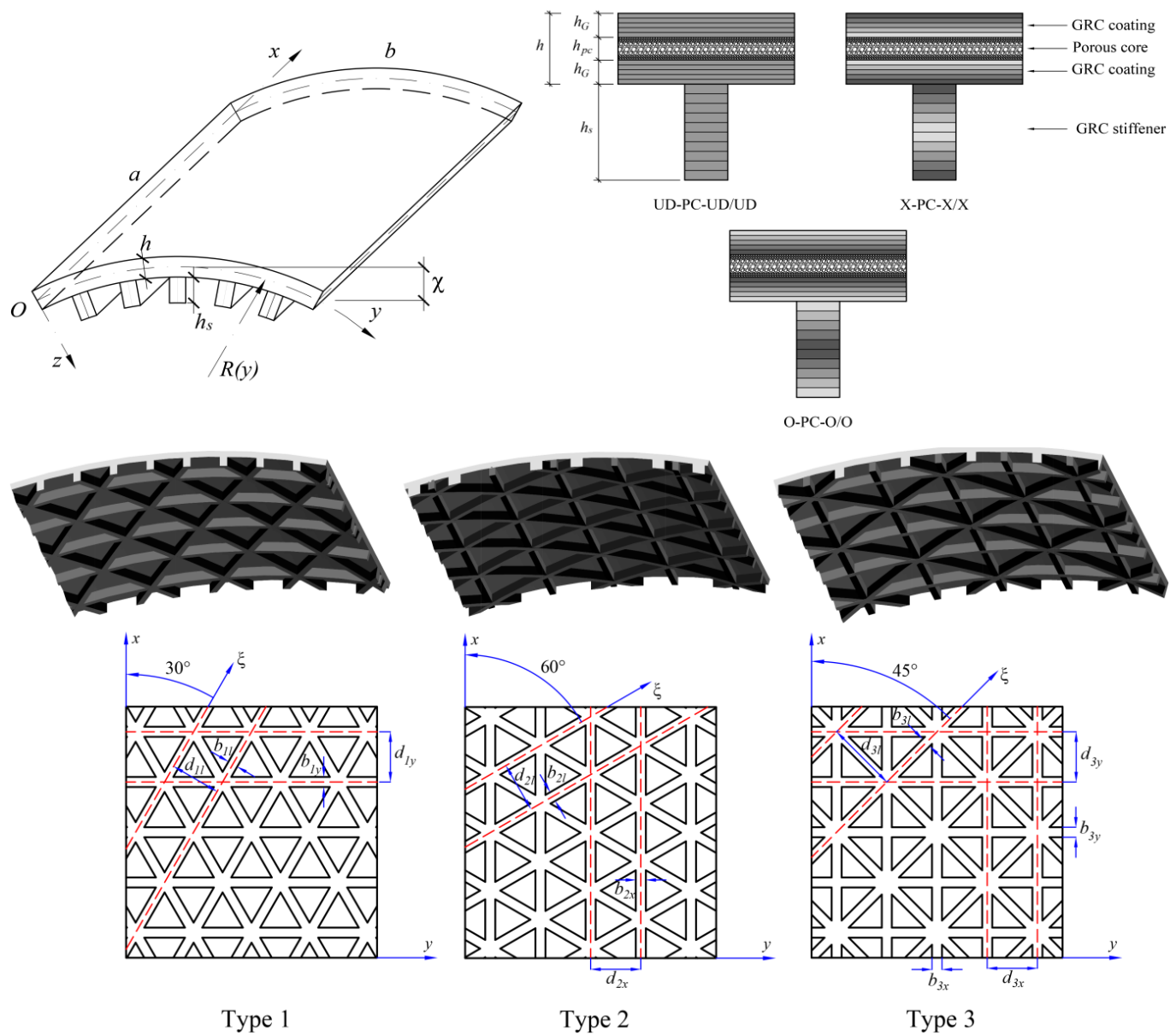


Fig. 1. Configurations of panels and three types of lattice stiffeners

The surface profile of the parabolic and sinusoidal panels in the circumferential direction is described by the following configuration functions, respectively

$$\begin{aligned} z_{[p]} &= -\frac{\chi y}{b} \left(\frac{4y}{b} - 4 \right), \\ z_{[s]} &= \chi \sin \left(\frac{\pi y}{b} \right). \end{aligned} \quad (2)$$

The corresponding radii of curvature for the parabolic and sinusoidal panels can be derived from Eq. (2) and are expressed as follows [35]

$$R_{[s]} = \frac{\left[\chi^2 \pi^2 \cos^2 \left(\frac{\pi y}{b} \right) + b^2 \right]^{1.5}}{b \chi \pi^2 \sin \left(\frac{\pi y}{b} \right)}, \quad (3a)$$

$$R_{[p]} = \frac{\left[b^4 + 16 \chi^2 (b - 2y)^2 \right]^{1.5}}{8 b^4 \chi}. \quad (3b)$$

The stress-strain relations for the panels, based on Hooke's law and incorporating the effect of uniform temperature variation, are expressed as follows

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11}^* & \bar{Q}_{12}^* & 0 \\ \bar{Q}_{12}^* & \bar{Q}_{22}^* & 0 \\ 0 & 0 & \bar{Q}_{66}^* \end{bmatrix} \begin{bmatrix} \varepsilon_x - \alpha_{11}\Delta T \\ \varepsilon_y - \alpha_{22}\Delta T \\ \gamma_{xy} \end{bmatrix}, \quad (4)$$

$$\sigma_{xz} = \bar{Q}_{44}^* \gamma_{xz}, \quad \sigma_{yz} = \bar{Q}_{55}^* \gamma_{yz},$$

where the uniform temperature rise ΔT is defined by the change of temperature from the free thermal state to the deformed state. The reduced stiffnesses for the GRC face sheets and GRC stiffeners can be obtained as

$$\bar{Q}_{11}^* = \frac{E_{11}}{1 - \nu_{12}\nu_{21}}, \bar{Q}_{12}^* = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}}, \bar{Q}_{22}^* = \frac{E_{22}}{1 - \nu_{12}\nu_{21}},$$

$$\bar{Q}_{66}^* = G_{12}, \bar{Q}_{44}^* = \bar{Q}_{55}^* = 0.5G_{12}.$$

for the porous core layer

$$\bar{Q}_{11}^* = \bar{Q}_{22}^* = \frac{E_{pc}}{1 - \nu_M^2}, \bar{Q}_{12}^* = \frac{\nu_M E_{pc}}{1 - \nu_M^2},$$

$$\bar{Q}_{44}^* = \bar{Q}_{55}^* = \bar{Q}_{66}^* = \frac{E_{pc}}{2(1 + \nu_M)}.$$

3. Governing equations

In this study, the panels are analyzed under the assumption of large deflection. The governing equations are formulated based on the HSDT, incorporating geometric nonlinearity and initial geometric imperfection. The relationships between strains, displacements, and rotations can be expressed as follows [33-35]

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \bar{\varepsilon}_x^{(0)} \\ \bar{\varepsilon}_y^{(0)} \\ \bar{\gamma}_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \bar{\varepsilon}_x^{(1)} \\ \bar{\varepsilon}_y^{(1)} \\ \bar{\gamma}_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \bar{\varepsilon}_x^{(3)} \\ \bar{\varepsilon}_y^{(3)} \\ \bar{\gamma}_{xy}^{(3)} \end{Bmatrix}, \quad (5)$$

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \bar{\gamma}_{xz}^{(0)} \\ \bar{\gamma}_{yz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \bar{\gamma}_{xz}^{(2)} \\ \bar{\gamma}_{yz}^{(2)} \end{Bmatrix},$$

where

$$\begin{Bmatrix} \bar{\varepsilon}_x^{(0)} \\ \bar{\varepsilon}_y^{(0)} \\ \bar{\gamma}_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} u_{,x} + w_{,x}^2/2 + w_{,x}\bar{w}_{,x}^* \\ v_{,y} + w_{,y}^2/2 + w_{,y}\bar{w}_{,y}^* - w/R(y) \\ v_{,x} + u_{,y} + w_{,x}w_{,y} + w_{,y}\bar{w}_{,x}^* + w_{,x}\bar{w}_{,y}^* \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{\varepsilon}_x^{(1)} \\ \bar{\varepsilon}_y^{(1)} \\ \bar{\gamma}_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ \phi_{y,x} + \phi_{x,y} \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{\varepsilon}_x^{(3)} \\ \bar{\varepsilon}_y^{(3)} \\ \bar{\gamma}_{xy}^{(3)} \end{Bmatrix} = -\frac{4}{3h^2} \begin{Bmatrix} w_{,xx} + \phi_{x,x} \\ w_{,yy} + \phi_{y,y} \\ \phi_{y,x} + 2w_{,xy} + \phi_{x,y} \end{Bmatrix}, \quad (6)$$

$$\begin{Bmatrix} \bar{\gamma}_{xz}^{(0)} \\ \bar{\gamma}_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} w_{,x} + \phi_x \\ w_{,y} + \phi_y \end{Bmatrix}, \quad \begin{Bmatrix} \bar{\gamma}_{xz}^{(2)} \\ \bar{\gamma}_{yz}^{(2)} \end{Bmatrix} = -\frac{4}{h^2} \begin{Bmatrix} w_{,x} + \phi_x \\ w_{,y} + \phi_y \end{Bmatrix}.$$

The relationships between the internal force and moment resultants and the corresponding strains, rotations, and transverse deflection for the three types of FG-GRC panels can be derived as follows

$$N_x = \bar{e}_{11}\bar{\varepsilon}_x^{(0)} + \bar{e}_{12}\bar{\varepsilon}_y^{(0)} + \bar{d}_{11}\bar{\varepsilon}_x^{(1)} + \bar{d}_{12}\bar{\varepsilon}_y^{(1)} + \bar{a}_{11}\bar{\varepsilon}_x^{(3)} + \bar{a}_{12}\bar{\varepsilon}_y^{(3)} - \Phi_{1x}\Delta T,$$

$$N_y = \bar{e}_{12}\bar{\varepsilon}_x^{(0)} + \bar{e}_{22}\bar{\varepsilon}_y^{(0)} + \bar{d}_{12}\bar{\varepsilon}_x^{(1)} + \bar{d}_{22}\bar{\varepsilon}_y^{(1)} + \bar{a}_{12}\bar{\varepsilon}_x^{(3)} + \bar{a}_{22}\bar{\varepsilon}_y^{(3)} - \Phi_{1y}\Delta T,$$

$$N_{xy} = \bar{e}_{66}\bar{\gamma}_{xy}^{(0)} + \bar{d}_{66}\bar{\gamma}_{xy}^{(1)} + \bar{a}_{66}\bar{\gamma}_{xy}^{(3)}, \quad (7a)$$

$$M_x = \bar{d}_{11}\bar{\varepsilon}_x^{(0)} + \bar{d}_{12}\bar{\varepsilon}_y^{(0)} + \bar{b}_{11}\bar{\varepsilon}_x^{(1)} + \bar{b}_{12}\bar{\varepsilon}_y^{(1)} + \bar{c}_{11}\bar{\varepsilon}_x^{(3)} + \bar{c}_{12}\bar{\varepsilon}_y^{(3)} - \Phi_{2x}\Delta T,$$

$$M_y = \bar{d}_{12}\bar{\varepsilon}_x^{(0)} + \bar{d}_{22}\bar{\varepsilon}_y^{(0)} + \bar{b}_{12}\bar{\varepsilon}_x^{(1)} + \bar{b}_{22}\bar{\varepsilon}_y^{(1)} + \bar{c}_{12}\bar{\varepsilon}_x^{(3)} + \bar{c}_{22}\bar{\varepsilon}_y^{(3)} - \Phi_{2y}\Delta T,$$

$$M_{xy} = \bar{d}_{66}\bar{\gamma}_{xy}^{(0)} + \bar{b}_{66}\bar{\gamma}_{xy}^{(1)} + \bar{c}_{66}\bar{\gamma}_{xy}^{(3)},$$

$$T_x = \bar{a}_{11}\bar{\varepsilon}_x^{(0)} + \bar{a}_{12}\bar{\varepsilon}_y^{(0)} + \bar{c}_{11}\bar{\varepsilon}_x^{(1)} + \bar{c}_{12}\bar{\varepsilon}_y^{(1)} + \bar{h}_{11}\bar{\varepsilon}_x^{(3)} + \bar{h}_{12}\bar{\varepsilon}_y^{(3)} - \Phi_{4x}\Delta T,$$

$$T_y = \bar{a}_{12}\bar{\varepsilon}_x^{(0)} + \bar{a}_{22}\bar{\varepsilon}_y^{(0)} + \bar{c}_{12}\bar{\varepsilon}_x^{(1)} + \bar{c}_{22}\bar{\varepsilon}_y^{(1)} + \bar{h}_{12}\bar{\varepsilon}_x^{(3)} + \bar{h}_{22}\bar{\varepsilon}_y^{(3)} - \Phi_{4y}\Delta T,$$

$$T_{xy} = \bar{a}_{66}\bar{\gamma}_{xy}^{(0)} + \bar{c}_{66}\bar{\gamma}_{xy}^{(1)} + \bar{h}_{66}\bar{\gamma}_{xy}^{(3)}.$$

The expressions of shear and the higher-order shear forces are written as

$$Q_x = \bar{f}_{44}\bar{\gamma}_{xz}^{(0)} + \bar{g}_{44}\bar{\gamma}_{xz}^{(2)}, Q_y = \bar{f}_{55}\bar{\gamma}_{yz}^{(0)} + \bar{g}_{55}\bar{\gamma}_{yz}^{(2)}, \quad (8)$$

$$S_x = \bar{g}_{44}\bar{\gamma}_{xz}^{(0)} + \bar{l}_{44}\bar{\gamma}_{xz}^{(2)}, S_y = \bar{g}_{55}\bar{\gamma}_{yz}^{(0)} + \bar{l}_{55}\bar{\gamma}_{yz}^{(2)}.$$

where the stiffnesses and thermal forces are presented by

$$(\bar{e}_{11}, \bar{d}_{11}, \bar{b}_{11}, \bar{a}_{11}, \bar{c}_{11}, \bar{h}_{11}) = (\bar{e}_{11}^p, \bar{d}_{11}^p, \bar{b}_{11}^p, \bar{a}_{11}^p, \bar{c}_{11}^p, \bar{h}_{11}^p)$$

$$+ \mu_1 \left[\frac{2b_{11}}{d_{11}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(30^\circ) \right]$$

$$\begin{aligned}
&(\bar{e}_{11}, \bar{d}_{11}, \bar{b}_{11}, \bar{a}_{11}, \bar{c}_{11}, \bar{h}_{11}) = (\bar{e}_{11}^p, \bar{d}_{11}^p, \bar{b}_{11}^p, \bar{a}_{11}^p, \bar{c}_{11}^p, \bar{h}_{11}^p) \\
&+ \mu_1 \left[\frac{2b_{1l}}{d_{1l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(30^\circ) \right] \\
&+ \mu_2 \left[\frac{b_{2x}}{d_{2x}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \right. \\
&\quad \left. + \frac{2b_{2l}}{d_{2l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(60^\circ) \right] \\
&+ \mu_3 \left[\frac{b_{3x}}{d_{3x}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \right. \\
&\quad \left. + \frac{2b_{3l}}{d_{3l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(45^\circ) \right], \\
&(\bar{e}_{22}, \bar{d}_{22}, \bar{b}_{22}, \bar{a}_{22}, \bar{c}_{22}, \bar{h}_{22}) = (\bar{e}_{22}^p, \bar{d}_{22}^p, \bar{b}_{22}^p, \bar{a}_{22}^p, \bar{c}_{22}^p, \bar{h}_{22}^p) \\
&+ \mu_2 \frac{2b_{2l}}{d_{2l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(60^\circ) \\
&+ \mu_1 \left[\frac{b_{1y}}{d_{1y}} (\bar{e}_{22}^{st}, \bar{d}_{22}^{st}, \bar{b}_{22}^{st}, \bar{a}_{22}^{st}, \bar{c}_{22}^{st}, \bar{h}_{22}^{st}) \right. \\
&\quad \left. + \frac{2b_{1l}}{d_{1l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(30^\circ) \right] \\
&+ \mu_3 \left[\frac{b_{3y}}{d_{3y}} (\bar{e}_{22}^{st}, \bar{d}_{22}^{st}, \bar{b}_{22}^{st}, \bar{a}_{22}^{st}, \bar{c}_{22}^{st}, \bar{h}_{22}^{st}) \right. \\
&\quad \left. + \frac{2b_{3l}}{d_{3l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \cos^4(45^\circ) \right], \\
&(\bar{e}_{12}, \bar{d}_{12}, \bar{b}_{12}, \bar{a}_{12}, \bar{c}_{12}, \bar{h}_{12}) = (\bar{e}_{12}^p, \bar{d}_{12}^p, \bar{b}_{12}^p, \bar{a}_{12}^p, \bar{c}_{12}^p, \bar{h}_{12}^p) \\
&+ \mu_1 \frac{2b_{1l}}{d_{1l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \times \\
&\quad \cos^2(30^\circ) \sin^2(30^\circ) \\
&+ \mu_2 \frac{2b_{2l}}{d_{2l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \times \\
&\quad \cos^2(60^\circ) \sin^2(60^\circ) \\
&+ \mu_3 \frac{2b_{3l}}{d_{3l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \times \\
&\quad \cos^2(45^\circ) \sin^2(45^\circ), \\
&(\bar{e}_{66}, \bar{d}_{66}, \bar{b}_{66}, \bar{a}_{66}, \bar{c}_{66}, \bar{h}_{66}) = (\bar{e}_{66}^p, \bar{d}_{66}^p, \bar{b}_{66}^p, \bar{a}_{66}^p, \bar{c}_{66}^p, \bar{h}_{66}^p) \\
&+ \mu_1 \frac{2b_{1l}}{d_{1l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \times \\
&\quad \cos^2(30^\circ) \sin^2(30^\circ) \\
&+ \mu_2 \frac{2b_{2l}}{d_{2l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \times
\end{aligned}$$

$$\begin{aligned}
&\cos^2(60^\circ) \sin^2(60^\circ) \\
&+ \mu_3 \frac{2b_{3l}}{d_{3l}} (\bar{e}_{11}^{st}, \bar{d}_{11}^{st}, \bar{b}_{11}^{st}, \bar{a}_{11}^{st}, \bar{c}_{11}^{st}, \bar{h}_{11}^{st}) \times \\
&\quad \cos^2(45^\circ) \sin^2(45^\circ), \\
&(\bar{f}_{44}, \bar{g}_{44}, \bar{l}_{44}) = (\bar{f}_{44}^p, \bar{g}_{44}^p, \bar{l}_{44}^p) \\
&+ \mu_1 \frac{2b_{1l}}{d_{1l}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) \sin^2(30^\circ) \\
&+ \mu_2 \left[\frac{b_{2x}}{d_{2x}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) + \frac{2b_{2l}}{d_{2l}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) \sin^2(60^\circ) \right] \\
&+ \mu_3 \left[\frac{b_{3x}}{d_{3x}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) + \frac{2b_{3l}}{d_{3l}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) \sin^2(45^\circ) \right], \\
&(\bar{f}_{44}, \bar{g}_{44}, \bar{l}_{44}) = (\bar{f}_{44}^p, \bar{g}_{44}^p, \bar{l}_{44}^p) \\
&+ \mu_1 \frac{2b_{1l}}{d_{1l}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) \sin^2(30^\circ) \\
&+ \mu_2 \left[\frac{b_{2x}}{d_{2x}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) + \frac{2b_{2l}}{d_{2l}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) \sin^2(60^\circ) \right] \\
&+ \mu_3 \left[\frac{b_{3x}}{d_{3x}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) + \frac{2b_{3l}}{d_{3l}} (\bar{f}_{44}^{st}, \bar{g}_{44}^{st}, \bar{l}_{44}^{st}) \sin^2(45^\circ) \right], \\
&(\Phi_{1x}, \Phi_{2x}, \Phi_{4x}) = (\Phi_{1x}^p, \Phi_{2x}^p, \Phi_{4x}^p) \\
&+ \mu_1 \frac{2b_{1l}}{d_{1l}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) \times \\
&\quad [\cos^6(30^\circ) + \sin^2(30^\circ) \cos^2(30^\circ)] \\
&+ \mu_2 \left\{ \frac{b_{2x}}{d_{2x}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) + \frac{2b_{2l}}{d_{2l}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) \times \right. \\
&\quad \left. [\cos^6(60^\circ) + \sin^2(60^\circ) \cos^2(60^\circ)] \right\} \\
&+ \mu_3 \left\{ \frac{b_{3x}}{d_{3x}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) + \frac{2b_{3l}}{d_{3l}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) \times \right. \\
&\quad \left. [\cos^6(45^\circ) + \sin^2(45^\circ) \cos^2(45^\circ)] \right\}, \\
&(\Phi_{1y}, \Phi_{2y}, \Phi_{4y}) = (\Phi_{1y}^p, \Phi_{2y}^p, \Phi_{4y}^p) \\
&+ \mu_2 \frac{2b_{2l}}{d_{2l}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) \times \\
&\quad [\sin^6(60^\circ) + \sin^2(60^\circ) \cos^2(60^\circ)] \\
&+ \mu_1 \left\{ \frac{b_{1y}}{d_{1y}} (\Phi_{1y}^{st}, \Phi_{2y}^{st}, \Phi_{4y}^{st}) + \frac{2b_{1l}}{d_{1l}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) \times \right. \\
&\quad \left. [\sin^6(30^\circ) + \sin^2(30^\circ) \cos^2(30^\circ)] \right\}
\end{aligned}$$

$$+\mu_3 \left\{ \frac{b_{3y}}{d_{3y}} (\Phi_{1y}^{st}, \Phi_{2y}^{st}, \Phi_{4y}^{st}) + \frac{2b_{3l}}{d_{3l}} (\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) \times \left[\sin^6(45^\circ) + \sin^2(45^\circ) \cos^2(45^\circ) \right] \right\}.$$

where, μ_1, μ_2 and μ_3 represent the type of stiffeners, $\mu_1 = 1, \mu_2 = 0, \mu_3 = 0$ are applied for type 1, $\mu_1 = 0, \mu_2 = 1, \mu_3 = 0$ are applied for type 2, $\mu_1 = 0, \mu_2 = 0, \mu_3 = 1$ are applied for type 3, and

$$\begin{aligned} &(\bar{e}_{ij}^p, \bar{d}_{ij}^p, \bar{b}_{ij}^p, \bar{a}_{ij}^p, \bar{c}_{ij}^p, \bar{h}_{ij}^p) = \\ &\int_{-h/2}^{h/2} \bar{Q}_{ij}^*(z) (1, z, z^2, z^3, z^4, z^6) dz, (i, j = 1, 2, 6), \\ &(\bar{f}_{ij}^p, \bar{g}_{ij}^p, \bar{l}_{ij}^p) = \int_{-h/2}^{h/2} \bar{Q}_{ij}^*(z) (1, z^2, z^4) dz, (i, j = 4, 5), \\ &(\Phi_{1x}^p, \Phi_{2x}^p, \Phi_{4x}^p) = \\ &\int_{-h/2}^{h/2} [\alpha_{11}(z) \bar{Q}_{11}^*(z) + \alpha_{22}(z) \bar{Q}_{12}^*(z)] (1, z, z^3) dz, \\ &(\Phi_{1y}^p, \Phi_{2y}^p, \Phi_{4y}^p) = \\ &\int_{-h/2}^{h/2} [\alpha_{11}(z) \bar{Q}_{12}^*(z) + \alpha_{22}(z) \bar{Q}_{22}^*(z)] (1, z, z^3) dz, \\ &\begin{bmatrix} \bar{e}_i^{st} & \bar{d}_i^{st} & \bar{a}_i^{st} \\ \bar{d}_i^{st} & \bar{b}_i^{st} & \bar{c}_i^{st} \\ \bar{a}_i^{st} & \bar{c}_i^{st} & \bar{h}_i^{st} \end{bmatrix} = \begin{bmatrix} e_i & d_i & a_i \\ d_i & b_i & c_i \\ a_i & c_i & h_i \end{bmatrix} \\ &- \begin{bmatrix} e_{12} & 0 & d_{12} & 0 & a_{12} & 0 \\ d_{12} & 0 & b_{12} & 0 & c_{12} & 0 \\ a_{12} & 0 & c_{12} & 0 & h_{12} & 0 \end{bmatrix} \times \\ &\begin{bmatrix} e_i & 0 & d_i & 0 & a_i & 0 \\ 0 & e_{66} & 0 & d_{66} & 0 & a_{66} \\ d_i & 0 & b_i & 0 & c_i & 0 \\ 0 & d_{66} & 0 & b_{66} & 0 & c_{66} \\ a_i & 0 & c_i & 0 & a_{66} & 0 \\ 0 & h_{22} & 0 & c_{66} & 0 & h_{66} \end{bmatrix}^{-1} \begin{bmatrix} e_{12} & d_{12} & a_{12} \\ 0 & 0 & 0 \\ d_{12} & b_{12} & c_{12} \\ a_{12} & c_{12} & h_{12} \\ 0 & 0 & 0 \end{bmatrix}, \\ &(i = 11, 22) \end{aligned}$$

$$(\mathbf{e}_{ij}, \mathbf{d}_{ij}, \mathbf{b}_{ij}, \mathbf{a}_{ij}, \mathbf{c}_{ij}, \mathbf{h}_{ij}) = \int_{h/2}^{h/2+h_s} \bar{Q}_{ij}^*(z) (1, z, z^2, z^3, z^4, z^6) dz, \\ (i, j = 1, 2, 6),$$

$$(\bar{f}_{ij}^{st}, \bar{g}_{ij}^{st}, \bar{l}_{ij}^{st}) = \int_{h/2}^{h/2+h_s} \bar{Q}_{ij}^*(z) (1, z^2, z^4) dz, (i, j = 4, 5),$$

$$(\Phi_{1x}^{st}, \Phi_{2x}^{st}, \Phi_{4x}^{st}) = \int_{h/2}^{h/2+h_s} (\bar{Q}_{11}^* \alpha_{11} + \bar{Q}_{12}^* \alpha_{22}) (1, z, z^3) dz, \\ (\Phi_{1y}^{st}, \Phi_{2y}^{st}, \Phi_{4y}^{st}) = \int_{h/2}^{h/2+h_s} (\bar{Q}_{12}^* \alpha_{11} + \bar{Q}_{22}^* \alpha_{22}) (1, z, z^3) dz.$$

Based on Eq. (6), the deformation compatibility equation is established as follows

$$\bar{\varepsilon}_{x,yy}^{(0)} + \bar{\varepsilon}_{y,xx}^{(0)} - \bar{\gamma}_{xy,xy}^{(0)} = -\frac{w_{,xx}}{R(y)} - \bar{w}_{,xx}^* w_{,yy} \\ + w_{,xy}^2 + 2w_{,xy} \bar{w}_{,xy}^* - w_{,xx} \bar{w}_{,yy}^* - w_{,xx} w_{,yy}^*. \quad (9)$$

The stress function $\hat{\phi}(x, y)$ is introduced as

$$N_x = \hat{\phi}_{,yy}, \quad N_{xy} = -\hat{\phi}_{,xy}, \quad N_y = \hat{\phi}_{,xx}. \quad (10)$$

By combining the expressions of the internal forces and moments in Eq. (7) with the condition given in Eq. (10), the deformation compatibility equation in Eq. (9) can be reformulated as follows

$$\begin{aligned} &\bar{e}_{22}^* \phi_{,xxxx} + (\bar{e}_{66}^* + 2\bar{e}_{12}^*) \phi_{,xxyy} + \bar{e}_{11}^* \phi_{,yyyy} \\ &+ \left(\bar{d}_{21}^* - \frac{4}{3h^2} \bar{a}_{21}^* \right) \phi_{,xxx} \\ &+ \left(\frac{4}{3h^2} \bar{a}_{66}^* - \frac{4}{3h^2} \bar{a}_{11}^* + \bar{d}_{11}^* - \bar{d}_{66}^* \right) \phi_{,x,xyy} \\ &+ \left(\frac{4}{3h^2} \bar{a}_{66}^* - \frac{4}{3h^2} \bar{a}_{22}^* + \bar{d}_{22}^* - \bar{d}_{66}^* \right) \phi_{,y,xyy} \\ &+ \left(\bar{d}_{12}^* - \frac{4}{3h^2} \bar{a}_{12}^* \right) \phi_{,yyy} - \frac{4}{3h^2} \bar{a}_{21}^* w_{,xxxx} \\ &- \frac{4}{3h^2} (\bar{a}_{11}^* + \bar{a}_{22}^* - 2\bar{a}_{66}^*) w_{,xxyy} - \frac{4}{3h^2} \bar{a}_{12}^* w_{,yyyy} \\ &+ \frac{w_{,xx}}{R(y)} + w_{,xx} \bar{w}_{,yy}^* + \bar{w}_{,xx}^* w_{,yy} \\ &- w_{,xy}^2 + w_{,xx} w_{,yy} - 2w_{,xy} \bar{w}_{,xy}^* = 0, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \bar{e}_{11}^* &= \bar{e}_{22} / (\bar{e}_{11} \bar{e}_{22} - \bar{e}_{12}^2), \quad \bar{e}_{12}^* = -\bar{e}_{12} / (\bar{e}_{11} \bar{e}_{22} - \bar{e}_{12}^2), \\ \bar{e}_{22}^* &= \bar{e}_{11} / (\bar{e}_{11} \bar{e}_{22} - \bar{e}_{12}^2), \quad \bar{e}_{66}^* = 1/\bar{e}_{66}, \\ \bar{d}_{11}^* &= \bar{e}_{11}^* \bar{d}_{11} + \bar{e}_{12}^* \bar{d}_{12}, \\ \bar{d}_{12}^* &= \bar{e}_{11}^* \bar{d}_{12} + \bar{e}_{12}^* \bar{d}_{22}, \quad \bar{d}_{21}^* = \bar{e}_{22}^* \bar{d}_{12} + \bar{e}_{12}^* \bar{d}_{11}, \\ \bar{d}_{22}^* &= \bar{e}_{22}^* \bar{d}_{22} + \bar{e}_{12}^* \bar{d}_{12}, \quad \bar{d}_{66}^* = \bar{e}_{66}^* \bar{d}_{66}, \\ \bar{a}_{11}^* &= \bar{e}_{11}^* \bar{a}_{11} + \bar{e}_{12}^* \bar{a}_{12}, \quad \bar{a}_{12}^* = \bar{e}_{11}^* \bar{a}_{12} + \bar{e}_{12}^* \bar{a}_{22}, \\ \bar{a}_{21}^* &= \bar{e}_{22}^* \bar{a}_{12} + \bar{e}_{12}^* \bar{a}_{11}, \quad \bar{a}_{22}^* = \bar{e}_{22}^* \bar{a}_{22} + \bar{e}_{12}^* \bar{a}_{12}, \\ \bar{a}_{66}^* &= \bar{e}_{66}^* \bar{a}_{66}. \end{aligned}$$

4. Boundary condition, solution and energy method

In this study, two types of boundary conditions are considered, both corresponding to simply supported panels along all four edges. For the first case, the edges are assumed to be simply supported with in-plane displacements unconstrained (4F), which can be expressed as follows

$$\begin{aligned} N_x = N_{0x} = 0, T_x = 0, M_x = 0, N_{xy} = 0, w = 0, \\ \phi_y = 0, \text{ at } x = 0, a, \\ N_y = N_{0y} = 0, T_y = 0, M_y = 0, N_{xy} = 0, w = 0, \\ \phi_x = 0, \text{ at } y = 0, b. \end{aligned} \quad (12)$$

In addition, a second boundary condition is considered in which all four edges are simply supported with in-plane displacements restrained (4I), expressed as follows

$$\begin{aligned} N_x = N_{0x}, T_x = 0, M_x = 0, u = 0, w = 0, \\ \phi_y = 0, \text{ at } x = 0, a, \\ N_y = N_{0y}, T_y = 0, M_y = 0, v = 0, w = 0, \\ \phi_x = 0, \text{ at } y = 0, b. \end{aligned} \quad (13)$$

To satisfy the boundary conditions specified in Eqs. (12) and (13), appropriate approximate solutions for the transverse deflection, rotations, and initial imperfection of the three panel configurations are assumed as follows [33-35]

$$\begin{aligned} w &= W \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ \bar{w}^* &= \zeta h \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ \phi_x &= \Phi_x \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ \phi_y &= \Phi_y \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \end{aligned} \quad (14)$$

where m and n are the buckling modes, ζ is the imperfect size, and W, Φ_x, Φ_y are the amplitudes of deflection and rotations, respectively.

The stress function is assumed in a form similar to that adopted for cylindrical panels, incorporating both linear and nonlinear components, and can be expressed as follows

$$\begin{aligned} \hat{\phi} &= \frac{1}{2} N_{0y} x^2 + \frac{1}{2} N_{0x} y^2 + \hat{\phi}_1 \cos\left(\frac{2m\pi x}{a}\right) \\ &+ \hat{\phi}_2 \cos\left(\frac{2n\pi y}{b}\right) + \hat{\phi}_3 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right). \end{aligned} \quad (15)$$

By substituting the assumed solutions in Eq. (14) and the stress function in Eq. (15) into Eq. (11), the Galerkin method is applied to the shallow panel system. As a result, the unknown coefficients in Eq. (15) can be determined as follows

$$\begin{aligned} \hat{\phi}_1 &= \bar{x}_{11}^* (W + 2\zeta h) W + \bar{x}_{12}^* W, \\ \hat{\phi}_2 &= \bar{x}_{21}^* (W + 2\zeta h) W + \bar{x}_{22}^* W, \\ \hat{\phi}_3 &= \bar{x}_{31}^* \Phi_x + \bar{x}_{32}^* \Phi_y + \bar{x}_{33}^* W, \end{aligned} \quad (16)$$

where the coefficients $\bar{x}_{ij}^*$ can be found in Appendix A.

The averaged expressions corresponding to the immovable edge conditions can be formulated as follows

$$\begin{aligned} \int_0^b \int_0^a u_{,x} dx dy &= \bar{y}_{41}^* W + \bar{y}_{42}^* \Phi_x + \bar{y}_{43}^* \Phi_y + \bar{y}_{45}^* N_{0x} \\ &+ \bar{y}_{44}^* W (2\zeta h + W) + \bar{y}_{46}^* N_{0y} + \bar{y}_{47}^* \Delta T = 0, \end{aligned} \quad (17)$$

$$\begin{aligned} \int_0^b \int_0^a v_{,y} dx dy &= \bar{y}_{51}^* W + \bar{y}_{52}^* \Phi_x + \bar{y}_{53}^* \Phi_y + \bar{y}_{55}^* N_{0x} \\ &+ \bar{y}_{54}^* W (2\zeta h + W) + \bar{y}_{56}^* N_{0y} + \bar{y}_{57}^* \Delta T = 0, \end{aligned} \quad (18)$$

where the coefficients $\bar{y}_{ij}^*$ can be found in the Appendix B.

The strain energy of the panel, the work performed by the external loads, and the kinetic energy of the system, including the effect of damping, are expressed as follows

$$\hat{Y}_{in} = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^b \int_0^a \begin{bmatrix} \sigma_{xz} \gamma_{xz} + \sigma_{xy} \gamma_{xy} + \sigma_{yz} \gamma_{yz} \\ + \sigma_x (\epsilon_x - \alpha_{11} \Delta T) \\ + \sigma_y (\epsilon_y - \alpha_{22} \Delta T) \end{bmatrix} dx dy dz. \quad (19)$$

$$\hat{Y}_{ext} = \int_0^b \int_0^a q w dx dy. \quad (20)$$

$$\hat{Y}_t = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^b \int_0^a \rho(z) (w_{,t} + \bar{w}_{,t}^*)^2 dx dy dz. \quad (21)$$

The Lagrangian of the system, representing the total energy of the panel, can be expressed as follows

$$\hat{Y}_{\text{Total}} = \hat{Y}_t - \hat{Y}_{\text{in}} + \hat{Y}_{\text{ext}}. \quad (22)$$

By employing the Rayleigh dissipation function, the damping potential is defined as follows

$$\hat{d}_v = \frac{1}{2} \int_0^b \int_0^a c w^2 dx dy. \quad (23)$$

where c is damping coefficient.

The governing equations of motion are derived by applying the Euler-Lagrange equations in conjunction with the Rayleigh dissipation function, and can be expressed as follows

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \hat{Y}_{\text{Total}}}{\partial \dot{W}} \right) - \frac{\partial \hat{Y}_{\text{Total}}}{\partial W} + \frac{\partial \hat{d}_v}{\partial \dot{W}} &= 0, \\ \frac{d}{dt} \left(\frac{\partial \hat{Y}_{\text{Total}}}{\partial \dot{\Phi}_x} \right) - \frac{\partial \hat{Y}_{\text{Total}}}{\partial \Phi_x} &= 0, \\ \frac{d}{dt} \left(\frac{\partial \hat{Y}_{\text{Total}}}{\partial \dot{\Phi}_y} \right) - \frac{\partial \hat{Y}_{\text{Total}}}{\partial \Phi_y} &= 0, \end{aligned} \quad (24)$$

leads to

$$\begin{aligned} \bar{y}_{11}^* W + \bar{y}_{12}^* \Phi_x + \bar{y}_{13}^* \Phi_y + (W + \zeta h) \\ \cdot [\bar{y}_{14}^* \Phi_x + \bar{y}_{15}^* \Phi_y + \bar{y}_{17}^* W (W + 2\zeta h) + \bar{y}_{19}^* N_{0x}] \end{aligned} \quad (25)$$

$$\begin{aligned} + \bar{y}_{16}^* W (W + 4/3 \zeta h) + \bar{y}_{18}^* N_{0x} + \bar{y}_{110}^* N_{0y} \\ - \bar{y}_{112}^* q - \bar{y}_{113}^* \Delta T + \bar{y}_{115}^* \ddot{W} + \bar{y}_{116}^* c \dot{W} = 0, \\ \bar{y}_{12}^* W + \bar{y}_{22}^* \Phi_x + \bar{y}_{23}^* \Phi_y + \bar{y}_{24}^* W (W + 2\zeta h) \\ + \bar{y}_{25}^* N_{0x} + \bar{y}_{26}^* N_{0y} - \bar{y}_{27}^* \Delta T = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \bar{y}_{13}^* W + \bar{y}_{23}^* \Phi_x + \bar{y}_{33}^* \Phi_y + \bar{y}_{34}^* W (W + 2\zeta h) \\ + \bar{y}_{35}^* N_{0x} + \bar{y}_{36}^* N_{0y} - \bar{y}_{37}^* \Delta T = 0, \end{aligned} \quad (27)$$

where the coefficients $\bar{y}_{ij}^*$ can be found in Appendix B.

The motion equation is obtained by using Eqs. (25-27), and combining with $N_{0x} = 0$ and $N_{0y} = 0$ for 4F boundary condition, with Eqs. (17) and (18), presented as

$$\begin{aligned} (\bar{\vartheta}_1 \bar{c}_{11}^* + \bar{\vartheta}_2 \bar{c}_{41}^*) W + (\bar{\vartheta}_1 \bar{c}_{110}^* + \bar{\vartheta}_2 \bar{c}_{46}^*) \Delta T \\ + (\zeta h + W) \times \\ [(\bar{\vartheta}_1 \bar{c}_{12}^* + \bar{\vartheta}_2 \bar{c}_{42}^*) \Delta T + (\bar{\vartheta}_1 \bar{c}_{13}^* + \bar{\vartheta}_2 \bar{c}_{43}^*) W] \\ + (2\zeta h + W) \times \\ [(\bar{\vartheta}_1 \bar{c}_{14}^* + \bar{\vartheta}_2 \bar{c}_{44}^*) W + (\bar{\vartheta}_1 \bar{c}_{15}^* + \bar{\vartheta}_2 \bar{c}_{45}^*) W (W + \zeta h)] \end{aligned}$$

$$\begin{aligned} + \bar{y}_{16}^* W (W + 4\zeta h/3) - \bar{y}_{112}^* q \\ + \bar{y}_{115}^* \ddot{W} + \bar{y}_{116}^* c \dot{W} = 0, \end{aligned} \quad (28)$$

where, $\bar{\vartheta}_1 = 1, \bar{\vartheta}_2 = 0$ for the case of the 4F condition, $\bar{\vartheta}_1 = 0, \bar{\vartheta}_2 = 1$ for the case of 4I, and

$$\begin{aligned} \bar{c}_{11}^* &= \bar{y}_{12}^* \bar{o}_{14}^* + \bar{y}_{13}^* \bar{o}_{24}^* + \bar{y}_{11}^*, \bar{c}_{12}^* = \bar{y}_{14}^* \bar{o}_{15}^* + \bar{y}_{15}^* \bar{o}_{25}^*, \\ \bar{c}_{13}^* &= \bar{y}_{14}^* \bar{o}_{14}^* + \bar{y}_{15}^* \bar{o}_{24}^*, \bar{c}_{14}^* = \bar{y}_{12}^* \bar{o}_{11}^* + \bar{y}_{15}^* \bar{o}_{24}^*, \\ \bar{c}_{15}^* &= \bar{y}_{14}^* \bar{o}_{11}^* + \bar{y}_{15}^* \bar{o}_{21}^* + \bar{y}_{17}^*, \bar{c}_{16}^* = \bar{y}_{12}^* \bar{o}_{13}^* + \bar{y}_{13}^* \bar{o}_{23}^* + \bar{y}_{18}^*, \\ \bar{c}_{17}^* &= \bar{y}_{14}^* \bar{o}_{13}^* + \bar{y}_{15}^* \bar{o}_{23}^* + \bar{y}_{19}^*, \bar{c}_{18}^* = \bar{y}_{12}^* \bar{o}_{12}^* + \bar{y}_{13}^* \bar{o}_{22}^* + \bar{y}_{110}^*, \\ \bar{c}_{19}^* &= \bar{y}_{14}^* \bar{o}_{12}^* + \bar{y}_{15}^* \bar{o}_{22}^*, \bar{c}_{110}^* = \bar{y}_{12}^* \bar{o}_{15}^* + \bar{y}_{13}^* \bar{o}_{25}^* - \bar{y}_{113}^*, \\ \bar{c}_{41}^* &= \bar{c}_{11}^* - \frac{\bar{c}_{16}^* (\bar{o}_{32}^* \bar{o}_{43}^* + \bar{o}_{42}^*) + \bar{c}_{18}^* (\bar{o}_{33}^* \bar{o}_{42}^* + \bar{o}_{32}^*)}{\bar{o}_{33}^* \bar{o}_{43}^* - 1}, \\ \bar{c}_{42}^* &= \bar{c}_{12}^* - \frac{\bar{c}_{17}^* (\bar{o}_{34}^* \bar{o}_{43}^* + \bar{o}_{44}^*) + \bar{c}_{19}^* (\bar{o}_{33}^* \bar{o}_{44}^* + \bar{o}_{34}^*)}{\bar{o}_{33}^* \bar{o}_{43}^* - 1}, \\ \bar{c}_{43}^* &= \bar{c}_{13}^* - \frac{\bar{c}_{17}^* (\bar{o}_{32}^* \bar{o}_{43}^* + \bar{o}_{42}^*) + \bar{c}_{19}^* (\bar{o}_{33}^* \bar{o}_{42}^* + \bar{o}_{32}^*)}{\bar{o}_{33}^* \bar{o}_{43}^* - 1}, \\ \bar{c}_{44}^* &= \bar{c}_{14}^* - \frac{\bar{c}_{16}^* (\bar{o}_{31}^* \bar{o}_{43}^* + \bar{o}_{41}^*) + \bar{c}_{18}^* (\bar{o}_{33}^* \bar{o}_{41}^* + \bar{o}_{31}^*)}{\bar{o}_{33}^* \bar{o}_{43}^* - 1}, \\ \bar{c}_{45}^* &= \bar{c}_{15}^* - \frac{\bar{c}_{17}^* (\bar{o}_{31}^* \bar{o}_{43}^* + \bar{o}_{41}^*) + \bar{c}_{19}^* (\bar{o}_{33}^* \bar{o}_{41}^* + \bar{o}_{31}^*)}{\bar{o}_{33}^* \bar{o}_{43}^* - 1}, \\ \bar{c}_{46}^* &= \bar{c}_{110}^* - \frac{\bar{c}_{16}^* (\bar{o}_{34}^* \bar{o}_{43}^* + \bar{o}_{44}^*) + \bar{c}_{18}^* (\bar{o}_{33}^* \bar{o}_{44}^* + \bar{o}_{34}^*)}{\bar{o}_{33}^* \bar{o}_{43}^* - 1}, \\ \bar{o}_{11}^* &= \frac{\bar{y}_{33}^* \bar{y}_{24}^* - \bar{y}_{34}^* \bar{y}_{23}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \bar{o}_{12}^* = -\frac{\bar{y}_{23}^* \bar{y}_{36}^* - \bar{y}_{33}^* \bar{y}_{26}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \\ \bar{o}_{13}^* &= -\frac{\bar{y}_{35}^* \bar{y}_{23}^* - \bar{y}_{33}^* \bar{y}_{25}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \bar{o}_{14}^* = -\frac{-\bar{y}_{33}^* \bar{y}_{12}^* + \bar{y}_{13}^* \bar{y}_{23}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \\ \bar{o}_{15}^* &= \frac{\bar{y}_{37}^* \bar{y}_{23}^* - \bar{y}_{33}^* \bar{y}_{27}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \bar{o}_{16}^* = \frac{\bar{y}_{38}^* \bar{y}_{23}^* - \bar{y}_{33}^* \bar{y}_{28}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \\ \bar{o}_{21}^* &= -\frac{-\bar{y}_{34}^* \bar{y}_{22}^* + \bar{y}_{23}^* \bar{y}_{24}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \bar{o}_{22}^* = \frac{\bar{y}_{36}^* \bar{y}_{22}^* - \bar{y}_{23}^* \bar{y}_{26}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \\ \bar{o}_{23}^* &= \frac{-\bar{y}_{23}^* \bar{y}_{25}^* + \bar{y}_{35}^* \bar{y}_{23}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \bar{o}_{24}^* = \frac{-\bar{y}_{12}^* \bar{y}_{23}^* + \bar{y}_{13}^* \bar{y}_{22}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \\ \bar{o}_{25}^* &= \frac{-\bar{y}_{22}^* \bar{y}_{37}^* + \bar{y}_{23}^* \bar{y}_{27}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \bar{o}_{26}^* = \frac{-\bar{y}_{22}^* \bar{y}_{38}^* + \bar{y}_{23}^* \bar{y}_{28}^*}{-\bar{y}_{22}^* \bar{y}_{33}^* + \bar{y}_{23}^{*2}}, \\ \bar{o}_{31}^* &= -\frac{\bar{y}_{52}^* \bar{o}_{11}^* + \bar{y}_{53}^* \bar{o}_{21}^* + \bar{y}_{54}^*}{\bar{y}_{52}^* \bar{o}_{12}^* + \bar{y}_{53}^* \bar{o}_{22}^* + \bar{y}_{56}^*}, \\ \bar{o}_{32}^* &= -\frac{\bar{y}_{52}^* \bar{o}_{14}^* + \bar{y}_{53}^* \bar{o}_{24}^* + \bar{y}_{51}^*}{\bar{y}_{52}^* \bar{o}_{12}^* + \bar{y}_{53}^* \bar{o}_{22}^* + \bar{y}_{56}^*}, \end{aligned}$$

$$\begin{aligned}
\bar{o}_{33}^* &= -\frac{\bar{y}_{52}^* \bar{o}_{13}^* + \bar{y}_{53}^* \bar{o}_{23}^* + \bar{y}_{55}^*}{\bar{y}_{52}^* \bar{o}_{12}^* + \bar{y}_{53}^* \bar{o}_{22}^* + \bar{y}_{56}^*}, \\
\bar{o}_{34}^* &= -\frac{\bar{y}_{52}^* \bar{o}_{15}^* + \bar{y}_{53}^* \bar{o}_{25}^* + \bar{y}_{57}^*}{\bar{y}_{52}^* \bar{o}_{12}^* + \bar{y}_{53}^* \bar{o}_{22}^* + \bar{y}_{56}^*}, \\
\bar{o}_{35}^* &= -\frac{\bar{y}_{52}^* \bar{o}_{16}^* + \bar{y}_{53}^* \bar{o}_{26}^* + \bar{y}_{58}^*}{\bar{y}_{52}^* \bar{o}_{12}^* + \bar{y}_{53}^* \bar{o}_{22}^* + \bar{y}_{56}^*}, \\
\bar{o}_{41}^* &= -\frac{\bar{o}_{11}^* \bar{y}_{42}^* + \bar{o}_{21}^* \bar{y}_{43}^* + \bar{y}_{44}^*}{\bar{o}_{13}^* \bar{y}_{42}^* + \bar{o}_{23}^* \bar{y}_{43}^* + \bar{y}_{45}^*}, \\
\bar{o}_{42}^* &= -\frac{\bar{o}_{14}^* \bar{y}_{42}^* + \bar{o}_{24}^* \bar{y}_{43}^* + \bar{y}_{41}^*}{\bar{o}_{13}^* \bar{y}_{42}^* + \bar{o}_{23}^* \bar{y}_{43}^* + \bar{y}_{45}^*}, \\
\bar{o}_{43}^* &= -\frac{\bar{o}_{12}^* \bar{y}_{42}^* + \bar{o}_{22}^* \bar{y}_{43}^* + \bar{y}_{46}^*}{\bar{o}_{13}^* \bar{y}_{42}^* + \bar{o}_{23}^* \bar{y}_{43}^* + \bar{y}_{45}^*}, \\
\bar{o}_{44}^* &= -\frac{\bar{o}_{15}^* \bar{y}_{42}^* + \bar{o}_{25}^* \bar{y}_{43}^* + \bar{y}_{47}^*}{\bar{o}_{13}^* \bar{y}_{42}^* + \bar{o}_{23}^* \bar{y}_{43}^* + \bar{y}_{45}^*}, \\
\bar{o}_{45}^* &= -\frac{\bar{o}_{16}^* \bar{y}_{42}^* + \bar{o}_{26}^* \bar{y}_{43}^* + \bar{y}_{48}^*}{\bar{o}_{13}^* \bar{y}_{42}^* + \bar{o}_{23}^* \bar{y}_{43}^* + \bar{y}_{45}^*}.
\end{aligned}$$

The governing equation for linear free vibration is obtained by neglecting the nonlinear terms and external loading components. Accordingly, the corresponding natural frequency can be determined as follows

$$\omega_{mn} = \sqrt{\frac{\bar{\vartheta}_1 \bar{\zeta}_{11}^* + \bar{\vartheta}_2 \bar{\zeta}_{41}^* + (\bar{\vartheta}_1 \bar{\zeta}_{12}^* + \bar{\vartheta}_2 \bar{\zeta}_{42}^*) \Delta T}{\bar{y}_{115}^*}}. \quad (29)$$

In this study, two types of impulsive loads are considered, namely a finite-duration step load and

a triangular blast load, which are defined by the following functions, respectively

$$q(t) = \begin{cases} \text{Triangular blast load} \\ \left\{ Q \left(1 - \frac{t}{T} \right), \quad 0 \leq t \leq T \right. \\ 0, \quad t > T \\ \text{Finite duration step load} \\ \left\{ Q, \quad 0 \leq t \leq T \right. \\ 0, \quad t > T \end{cases} \quad (30)$$

where $T = \frac{2\pi}{\omega_{mn}}$ is the vibration period.

It should be noted that a blast load is characterized by a rapidly varying transient profile with a peak followed by decay, whereas a step load is defined as a constant load applied over a finite duration, leading to different temporal energy input and dynamic response characteristics.

By substituting the impulsive load functions from Eq. (30) into Eq. (28), the resulting governing equation is solved using the Runge-Kutta method to obtain the dynamic responses of the panels. The initial conditions are assumed to correspond to a state of rest, with both the initial deflection and initial velocity taken as zero, as

$$W|_{t=0} = 0, \quad \dot{W}|_{t=0} = 0, \quad (31)$$

5. Results and discussions

Table 1. Validation of the fundamental frequency parameters $\tilde{\omega}_{mn} = \omega_{mn} (a^2/h) \sqrt{\rho_0/E_0}$

of (0/90/0/90/0)_s FG-GRC cylindrical panels (4F, $a/b = 1$, $h = 2$ mm, $a/R_{[c]} = 0.5$, $b/h = 20$, $m = n = 1$)

T (K)	Distribution	Shen et al. [27]	Present
300	UD	38.0071	38.0804
	FG-X	39.5231	39.6005
	FG-V	34.6249	34.6964
	FG-Λ	36.2314	36.3017
	FG-O	33.8763	33.9400
400	UD	35.3452	35.4113
	FG-X	36.4831	36.5527
	FG-V	32.7796	32.8433
	FG-Λ	33.3561	33.1578
	FG-O	31.4769	31.5344

Table 2. Effects of graphene distribution laws, panel types and stiffener types on the fundamental frequencies (rad/s) of FG-GRC panels (4F, $\chi = 2h$, $\zeta = 0$)

Type	Panel	UD-PC-UD/UD	X-PC-X/X	O-PC-O/O
Type 1	Sinusoid	82531.07	80924.49	76703.86
	Parabolic	82523.45	80917.57	76696.59
	Cylindrical	82522.73	80917.06	76696.05
Type 2	Sinusoid	81857.13	80355.98	76164.67
	Parabolic	81848.53	80347.03	76155.25
	Cylindrical	81847.67	80346.20	76154.38
Type 3	Sinusoid	79499.68	78151.40	73574.43
	Parabolic	79456.36	78111.10	73531.62
	Cylindrical	79452.53	78107.66	73527.97

Table 1 presents a comparison of the fundamental frequency parameters of FG-GRC cylindrical panels obtained from the present study and those reported by Shen et al. [27] under different graphene distribution patterns and temperature conditions. It can be observed that the results from the present formulation are in very close agreement with the reference data for all considered cases. The discrepancies are negligible, indicating the high accuracy and reliability of the proposed semi-analytical model.

The geometric, temperature, and damping parameters used for the panels and stiffeners are specified as follows: $h=2\text{mm}$, $h_c=0.8h$, $a=b=20h$, $e_0=0.3$, $\chi=0.5h$, $\zeta=0.1$, $T=400\text{K}$, $m=n=1$, $b_{1y}=b_{1l}=h$, $b_{2x}=b_{2l}=h$, $b_{3x}=b_{3y}=b_{3l}=h$, $h_s=1.5h$, $d_{1y}=d_{1l}=a/8$, $d_{2x}=d_{2l}=a/8$, $d_{3x}=d_{3y}=a/8$, $d_{3l}=\sqrt{2}a/8$. It should be noted that these parameters may be modified or updated as required by the numerical analyses. The material properties of the FG-GRC layers and porous core, including Young's moduli, density, Poisson's ratios, and thermal expansion coefficients, are adopted from Ref. [27].

Table 2 presents the effects of graphene distribution patterns, panel geometries, and lattice stiffener types on the fundamental frequencies of FG-GRC panels. It is clearly observed that the

graphene distribution has a significant influence on the dynamic characteristics of the structure. Among the considered cases, the UD-PC-UD/UD configuration consistently yields the highest natural frequencies, followed by the X-PC-X/X and O-PC-O/O distributions. This trend indicates that a more uniform distribution of graphene across the thickness leads to higher overall stiffness, while the FG-O pattern, with reduced reinforcement near the outer surfaces, results in the lowest stiffness and frequency values. Regarding the effect of stiffener types, Type 1 provides the highest frequencies, whereas Type 3 results in the lowest values for all panel configurations. This behavior demonstrates that the stiffener topology plays a crucial role in enhancing structural rigidity, with more effective stiffening layouts significantly improving the dynamic performance. In contrast, the influence of panel geometry is relatively minor compared to material distribution and stiffener configuration. The differences in natural frequencies among sinusoidal, parabolic, and cylindrical panels are negligible, indicating that the curvature variation has a limited effect on the fundamental frequency under the considered shallow shell conditions. Table 3 illustrates the combined effects of temperature and stiffener spacing on the fundamental frequencies of FG-GRC sinusoidal panels. It is evident that temperature has a pronounced influence on the dynamic behavior. As

the temperature increases from 300 K to 500 K, the natural frequency consistently decreases for all stiffener types and spacing configurations. This reduction can be attributed to the degradation of material stiffness at elevated temperatures, which weakens the overall structural rigidity. In contrast, decreasing the stiffener spacing leads to a significant increase in the natural frequency. This trend indicates that the corresponding stiffener configuration enhances the bending stiffness of the

panel, thereby improving its dynamic performance. The increase is substantial across all cases, confirming that stiffener spacing is a dominant design parameter. Furthermore, the influence of stiffener type is consistently observed, with Type 3 yielding the highest frequencies, while Types 1 and 2 exhibit very close values. This behavior suggests that the structural efficiency of the stiffener layout strongly affects the load-carrying capacity and stiffness enhancement.

Table 3. Effects of temperature changes and stiffener distances on the fundamental frequencies (rad/s) of FG-GRC sinusoid panels (UD-PC-UD/UD, 4I, $\chi = 2h$, $\zeta = 0$, $d_{1y} = d_{1l} = d_{2x} = d_{2l} = d_{3x} = d_{3y} = d_s$, $d_{3l} = \sqrt{2}d_s$)

	T (K)	300	400	500
$d_s = a/4$	Type 1	113403.39	105771.70	102459.47
	Type 2	113509.99	105792.15	102354.55
	Type 3	115411.84	107714.54	104537.29
$d_s = a/8$	Type 1	123622.73	115515.73	112208.18
	Type 2	123984.02	115769.99	112362.68
	Type 3	126314.67	118097.00	114919.10
$d_s = a/12$	Type 1	128128.58	119803.99	116497.18
	Type 2	128597.56	120150.31	116770.69
	Type 3	131139.25	122678.39	119489.00

Fig. 2 presents the effects of panel type, stiffener type, and edge length on the time-history deflection responses of FG-GRC panels under blast and step loads. In all subfigures, the response can be divided into two stages, namely the forced-vibration stage during load application and the free-vibration stage after load removal. A clear distinction between these two stages is observed in both loading cases.

As shown in Figs. 2(a) and 2(b), the effect of panel geometry on the dynamic response is relatively small during the forced-vibration stage, since the three panel types exhibit nearly identical peak deflections while the external load is acting. However, noticeable differences appear in the subsequent free-vibration stage. The sinusoidal panel generally shows the largest oscillation amplitude, whereas the cylindrical panel exhibits

the smallest one, with the parabolic panel lying in between.

The influence of stiffener type is illustrated in Figs. 2(c) and 2(d). It is evident that the three stiffener layouts lead to different deflection amplitudes, although the differences are not very large compared with those caused by edge length. Among the considered cases, Type 3 tends to produce the largest oscillation amplitude, while Type 1 provides the smallest response.

A much more significant effect is observed in Figs. 2(e) and 2(f), where the edge length is varied. Increasing the panel size results in a substantial increase in the deflection amplitude under both blast and step loads. Larger panels possess lower global bending stiffness and are therefore more vulnerable to dynamic deformation. This effect is especially evident in the free-vibration stage,

where the oscillation amplitudes increase markedly with increasing edge length. A further comparison between blast and step loading reveals that the step load generally induces larger peak responses than the blast load. This is because the step load

maintains a constant magnitude over a finite duration, allowing more energy to be transferred into the structure, whereas the blast load decays with time and therefore introduces a shorter excitation.

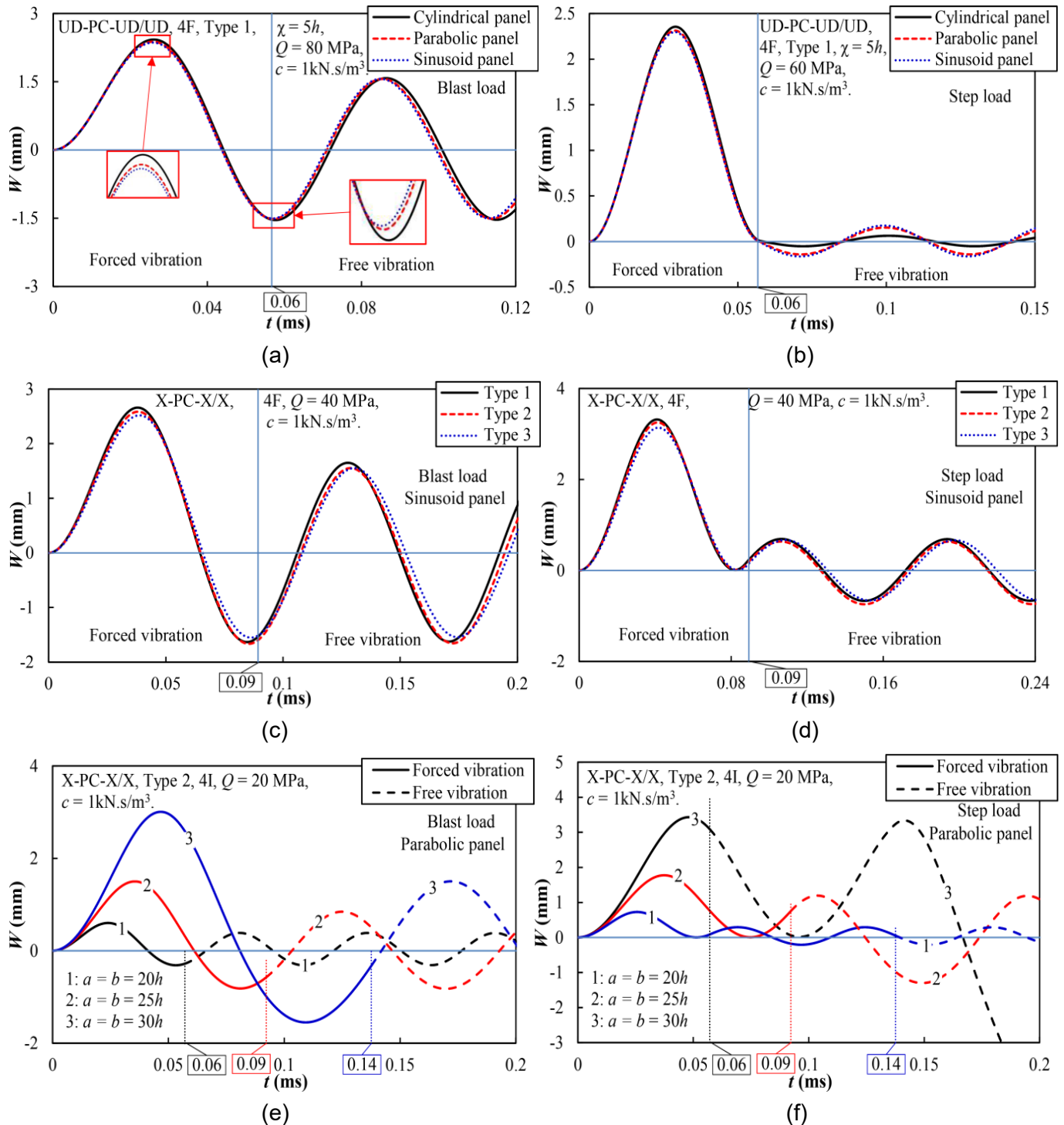


Fig. 2. Effects of panel types, stiffener types, and edge length on the dynamic responses of FG-GRC panels. (a) Effects of panel types (blast load); (b) Effects of panel types (step load); (c) Effects of stiffener types (blast load); (d) Effects of stiffener types (step load); (e) Effects of edge lengths (blast load); (f) Effects of edge lengths (step load)

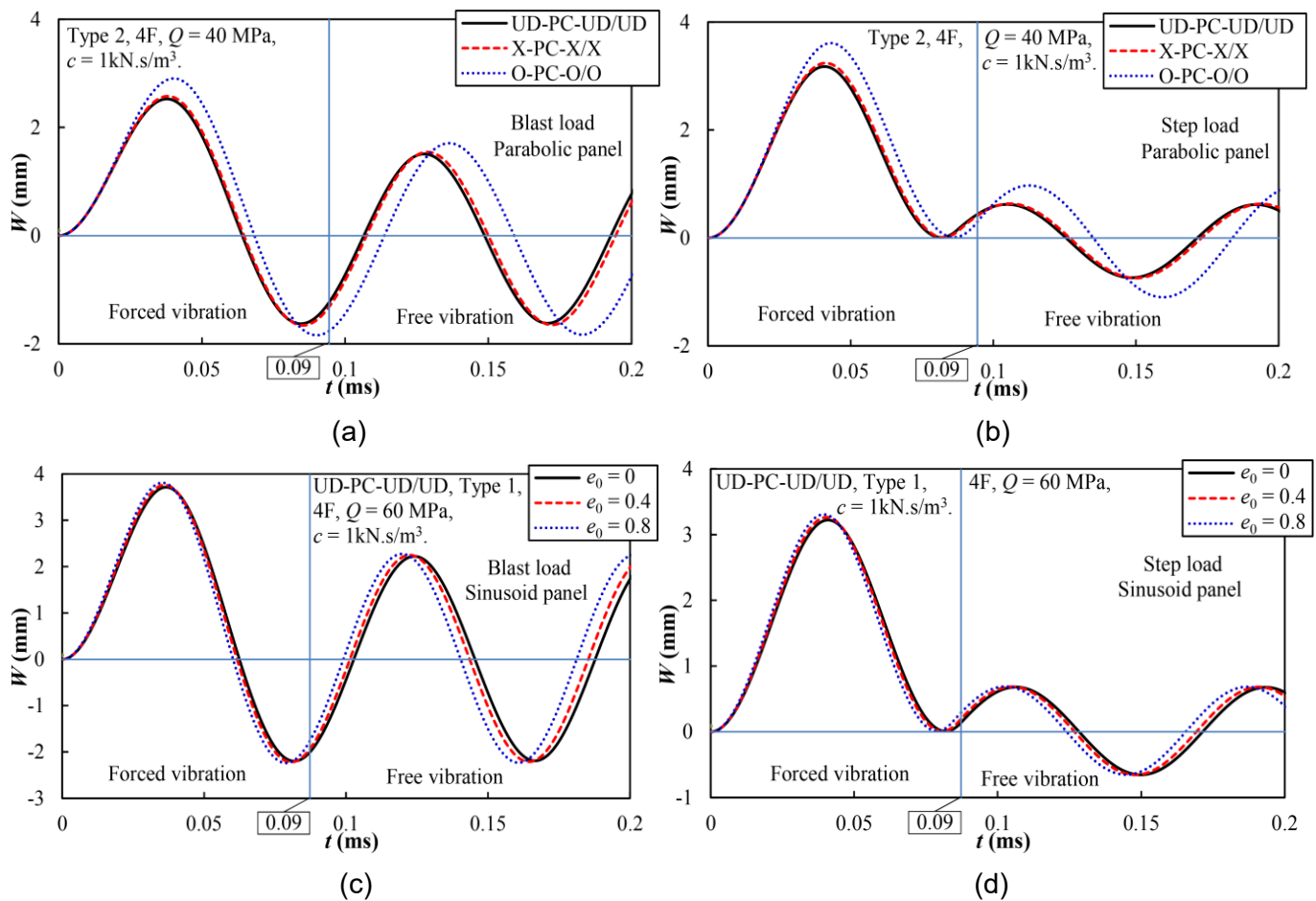


Fig. 3. Effects of graphene distribution laws and porosity coefficients on the dynamic responses of FG-GRC panels. (a) Effects of graphene distribution laws (blast load), (b) Effects of graphene distribution laws (step load), (c) Effects of porosity coefficients (blast load), (d) Effects of porosity coefficients (step load)

Fig. 3 illustrates the effects of graphene distribution laws and porosity coefficients on the dynamic responses of FG-GRC panels under blast and step loads. In all cases, the time-history response can be divided into a forced-vibration stage, during which the external load is applied, and a free-vibration stage after load removal. The differences among the considered configurations become more evident in the post-loading stage, where the oscillatory behavior is governed primarily by the structural stiffness.

As shown in Figs. 3(a) and 3(b), the graphene distribution law has a noticeable influence on the dynamic response of the panel. Among the three considered distributions, the O-PC-O/O configuration generally exhibits the largest deflection amplitude, whereas the UD-PC-UD/UD

case gives the smallest response, with X-PC-X/X lying in between. This trend indicates that the distribution of graphene through the thickness significantly affects the flexural rigidity of the structure. A more favorable reinforcement pattern enhances the stiffness of the panel and therefore suppresses the vibration amplitude, while a less efficient distribution leads to larger dynamic deflections. The same tendency is observed under both blast and step loads, although the differences become slightly more pronounced in the free-vibration stage.

The influence of the porosity coefficient is presented in Figs. 3(c) and 3(d). It is evident that increasing the porosity coefficient leads to larger deflection amplitudes under both loading conditions. In particular, the case with $e_0 = 0.8$

produces the largest response, while the non-porous core case ($e_0 = 0$) shows the smallest amplitude. This behavior is physically expected, since a higher porosity level reduces the effective

stiffness of the core layer and weakens the overall resistance of the sandwich panel to dynamic excitation. As a result, the structure becomes more compliant and exhibits stronger oscillations.

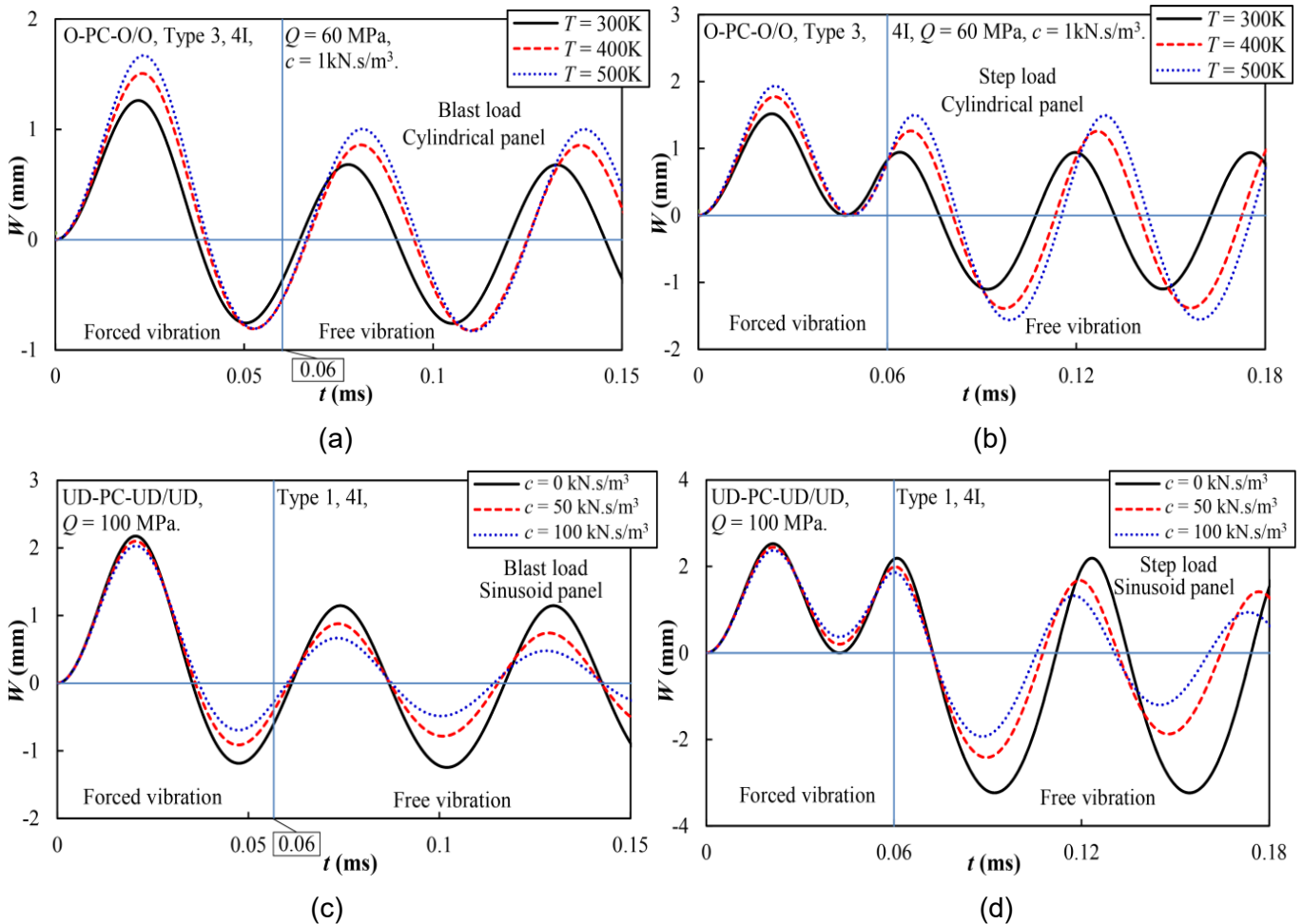


Fig. 4. Effects of temperatures and damping coefficients on the dynamic responses of FG-GRC panels. (a) Effects of temperatures (blast load); (b) Effects of temperatures (step load); (c) Effects of damping coefficients (blast load); (d) Effects of damping coefficients (step load)

Fig. 4 presents the effects of temperature and damping coefficient on the dynamic responses of FG-GRC panels under blast and step loads. In all cases, the response consists of a forced-vibration stage during load application and a free-vibration stage after the load is removed. The influence of both parameters becomes more evident in the post-loading stage, where the oscillation amplitude and decay rate are governed mainly by the structural stiffness and energy dissipation capability.

As shown in Figs. 4(a) and 4(b), temperature has a noticeable effect on the dynamic behavior of

the panel. When the temperature increases from 300 K to 500 K, the deflection amplitude increases under both blast and step loads. This trend indicates that elevated temperature reduces the effective stiffness of the FG-GRC panel, thereby making the structure more compliant under impulsive excitation. The influence of temperature is especially apparent in the free-vibration stage, where higher temperatures lead to larger oscillation amplitudes. Compared with the blast-load case, the temperature effect under step loading becomes more pronounced due to the greater amount of energy transferred into the

structure during the loading phase.

The effect of the damping coefficient is illustrated in Figs. 4(c) and 4(d). It is observed that increasing the damping coefficient significantly suppresses the vibration amplitude and accelerates the decay of oscillations after load removal. For both loading types, the case with zero damping exhibits the largest and most persistent oscillations, whereas higher damping coefficients lead to a much faster attenuation of the dynamic response.

6. Conclusion

This study has presented a semi-analytical investigation on the nonlinear dynamic behavior of lattice-stiffened variable-curvature FG-GRC sandwich panels with a porous core subjected to impulsive loading conditions. The formulation was developed based on higher-order shear deformation theory (HSDT) incorporating von Kármán geometric nonlinearity, and an improved smeared stiffener technique was employed to account for the effects of different lattice configurations. The governing equations were derived using the Lagrange approach and solved via a Runge-Kutta method to obtain the time-dependent responses.

The numerical results demonstrate that graphene distribution, stiffener topology, core porosity, geometric configuration, temperature, damping, and loading type all have significant influences on the dynamic behavior of the panels. In particular, a more effective graphene distribution enhances structural stiffness and reduces vibration amplitudes, while increasing porosity leads to a noticeable reduction in stiffness and larger deflections. The presence of lattice stiffeners significantly improves the dynamic performance, with appropriate stiffener configurations effectively increasing the natural frequency and suppressing oscillations.

It is also found that geometric parameters such as panel type and edge length influence the response, with larger panels exhibiting greater

deflections due to reduced global stiffness. Moreover, temperature rise results in stiffness degradation and increased vibration amplitudes, whereas higher damping coefficients effectively attenuate the oscillations. A comparison between loading cases indicates that step loads generally induce larger and more sustained responses than blast loads due to longer energy input duration.

The present study provides useful insights for the design and optimization of advanced composite structures subjected to extreme dynamic and thermal environments.

Declaration of Generative AI

During the preparation of this work, the authors did not use Generative AI and AI-assisted technology tools in order to generate the content of the manuscript.

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Appendix A

$$\bar{x}_{11}^* = \frac{\bar{n}^2}{32\bar{m}^2\bar{e}_{22}^*}, \bar{x}_{21}^* = \frac{\bar{m}^2}{32\bar{n}^2\bar{e}_{11}^*}, \bar{\ell} = [(-1)^m - 1][(-1)^n - 1], \bar{m} = \frac{m\pi}{a}, \bar{n} = \frac{n\pi}{b},$$

$$\bar{x}_{12}^* = \frac{8\bar{\ell}^2 [\bar{R}_{2[c,p,s]} - \bar{R}_{3[c,p,s]}]}{72a^3b^3\bar{m}^6\bar{n}^2\bar{e}_{22}^* - 128ab\bar{m}^4\bar{e}_{22}^*\bar{\ell}^2} - \frac{12\bar{m}\bar{n}ab\bar{\ell}\bar{R}_{1[c,p,s]}}{72a^3b^3\bar{m}^6\bar{n}^2\bar{e}_{22}^* - 128ab\bar{m}^4\bar{e}_{22}^*\bar{\ell}^2} - \frac{9\bar{R}_{2[c,p,s]}a^2b^2\bar{m}^2\bar{n}^2}{72a^3b^3\bar{m}^6\bar{n}^2\bar{e}_{22}^* - 128ab\bar{m}^4\bar{e}_{22}^*\bar{\ell}^2},$$

$$\bar{x}_{22}^* = \frac{-8\bar{\ell}^2 [\bar{R}_{2[c,p,s]} - \bar{R}_{3[c,p,s]}]}{72a^3b^3\bar{m}^2\bar{n}^6\bar{e}_{11}^* - 128ab\bar{n}^4\bar{e}_{11}^*\bar{\ell}^2} - \frac{12\bar{m}\bar{n}ab\bar{\ell}\bar{R}_{1[c,p,s]}}{72a^3b^3\bar{m}^2\bar{n}^6\bar{e}_{11}^* - 128ab\bar{n}^4\bar{e}_{11}^*\bar{\ell}^2} - \frac{9\bar{R}_{3[c,p,s]}a^2b^2\bar{m}^2\bar{n}^2}{72a^3b^3\bar{m}^2\bar{n}^6\bar{e}_{11}^* - 128ab\bar{n}^4\bar{e}_{11}^*\bar{\ell}^2},$$

$$\bar{x}_{31}^* = -\frac{\bar{m} \left[(\bar{a}_{66}^* - \bar{a}_{11}^*) \frac{4}{3h^2} + \bar{d}_{11}^* - \bar{d}_{66}^* \right] \bar{n}^2}{\bar{m}^4\bar{e}_{22}^* + \bar{n}^2(2\bar{e}_{12}^* + \bar{e}_{66}^*)\bar{m}^2 + \bar{n}^4\bar{e}_{11}^*} - \frac{\bar{m}^2 \left(-\frac{4}{3h^2}\bar{a}_{21}^* + \bar{d}_{21}^* \right)}{\bar{m}^4\bar{e}_{22}^* + \bar{n}^2(2\bar{e}_{12}^* + \bar{e}_{66}^*)\bar{m}^2 + \bar{n}^4\bar{e}_{11}^*},$$

$$\bar{x}_{32}^* = -\frac{\bar{n} \left\{ \left[(\bar{a}_{66}^* - \bar{a}_{22}^*) \frac{4}{3h^2} + \bar{d}_{22}^* - \bar{d}_{66}^* \right] \bar{m}^2 \right\}}{\bar{m}^4\bar{e}_{22}^* + \bar{n}^2(2\bar{e}_{12}^* + \bar{e}_{66}^*)\bar{m}^2 + \bar{n}^4\bar{e}_{11}^*} - \frac{\bar{n}^2 \left(-\frac{4}{3h^2}\bar{a}_{12}^* + \bar{d}_{12}^* \right)}{\bar{m}^4\bar{e}_{22}^* + \bar{n}^2(2\bar{e}_{12}^* + \bar{e}_{66}^*)\bar{m}^2 + \bar{n}^4\bar{e}_{11}^*},$$

$$\bar{x}_{33}^* = \frac{\frac{4}{3h^2}\bar{m}^4\bar{a}_{21}^* + \frac{4}{3h^2}\bar{n}^2(\bar{a}_{11}^* + \bar{a}_{22}^* - 2\bar{a}_{66}^*)\bar{m}^2 + \frac{4}{3h^2}\bar{n}^4\bar{a}_{12}^*}{\bar{m}^4\bar{e}_{22}^* + \bar{n}^2(2\bar{e}_{12}^* + \bar{e}_{66}^*)\bar{m}^2 + \bar{n}^4\bar{e}_{11}^*} - \frac{\bar{m}\bar{n} \left\{ 36ab\bar{m}\bar{n}\bar{R}_{1[c,p,s]} + 24\bar{\ell} [\bar{R}_{2[c,p,s]} + \bar{R}_{3[c,p,s]}] \right\}}{\left[\bar{m}^4\bar{e}_{22}^* + \bar{m}^2\bar{n}^2(2\bar{e}_{12}^* + \bar{e}_{66}^*) + \bar{n}^4\bar{e}_{11}^* \right] \{ 9a^2b^2\bar{m}^2\bar{n}^2 - 16\bar{\ell}^2 \}},$$

$$\bar{R}_{1[c]} = -\frac{2ab\chi\bar{m}^2}{4\chi^2 + b^2}, \bar{R}_{2[c]} = \bar{R}_{3[c]} = \frac{8\chi\bar{m}\bar{\ell}}{3\bar{n}(4\chi^2 + b^2)},$$

$$\bar{R}_{1[p,s]} = -\frac{1}{2}\bar{m}^2a \int_0^b \frac{\sin^2(\bar{n}y)}{R_{[p,s]}} dy, \bar{R}_{2[p,s]} = -\frac{1}{3}\bar{m} [(-1)^m - 1] \int_0^b \frac{\sin(\bar{n}y)}{R_{[p,s]}} dy,$$

$$\bar{R}_{3[p,s]} = \bar{m} [(-1)^m - 1] \int_0^b \frac{\sin(\bar{n}y) \cos(2\bar{n}y)}{R_{[p,s]}} dy.$$

Appendix B

$$\bar{y}_{11}^* = \frac{ab}{4} \left\{ \begin{aligned} & \left[32\bar{e}_{22}^* \bar{x}_{12}^{*2} + \bar{e}_{22}^* \bar{x}_{33}^{*2} + \left(\frac{4}{3h^2} \right)^2 \bar{h}_{11}^* \right] \bar{m}^4 + \left[\bar{l}_{44} \left(\frac{12}{3h^2} \right)^2 - \bar{g}_{44} \frac{8}{3h^2} + \bar{f}_{44} \right] \bar{m}^2 \\ & + \left[(\bar{h}_{12}^* + \bar{h}_{21}^* + 4\bar{h}_{66}^*) \left(\frac{4}{3h^2} \right)^2 + \bar{x}_{33}^{*2} (2\bar{e}_{12}^* + \bar{e}_{66}^*) \right] \bar{m}^2 \bar{n}^2 \\ & + \left[32\bar{e}_{11}^* \bar{x}_{22}^{*2} + \bar{e}_{11}^* \bar{x}_{33}^{*2} + \left(\frac{4}{3h^2} \right)^2 \bar{h}_{22}^* \right] \bar{n}^4 + \left[\bar{l}_{55} \left(\frac{12}{3h^2} \right)^2 - 6\bar{g}_{55} \frac{4}{3h^2} + \bar{f}_{55} \right] \bar{n}^2 \\ & - \frac{32[\bar{x}_{12}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 \bar{e}_{12}^* (\bar{x}_{12}^* + \bar{x}_{22}^*)]}{3ab\bar{m}\bar{n}} \bar{\ell} \bar{x}_{33}^* - \frac{\bar{n}^4 \bar{x}_{22}^* \bar{e}_{11}^* \bar{\ell} \bar{x}_{33}^*}{3ab\bar{m}\bar{n}} \end{aligned} \right\},$$

$$\bar{y}_{12}^* = \frac{ab}{8} \left(2\bar{x}_{31}^* \bar{x}_{33}^* \bar{m}^4 \bar{e}_{22}^* + \bar{n}^2 (4\bar{e}_{12}^* + 2\bar{e}_{66}^*) \bar{x}_{33}^* \bar{x}_{31}^* \bar{m}^2 + 2\bar{x}_{31}^* \bar{x}_{33}^* \bar{n}^4 \bar{e}_{11}^* \right) +$$

$$\frac{ab}{8} \left\{ \begin{aligned} & \left[(\bar{h}_{12}^* + \bar{h}_{21}^* + 4\bar{h}_{66}^*) \bar{n}^2 \left(\frac{4}{3h^2} \right)^2 - (\bar{c}_{21}^* + 2\bar{c}_{66}^* + \bar{c}_{12}^* + 2\bar{c}_{66}^*) \bar{n}^2 \frac{4}{3h^2} \right] \bar{m} \\ & + \left[18 \left(\frac{4}{3h^2} \right)^2 \bar{l}_{44} - 12\bar{g}_{44} \frac{4}{3h^2} + 2\bar{f}_{44} \right] \bar{m} + \frac{4}{3h^2} \left(\frac{8}{3h^2} \bar{h}_{11}^* - \bar{c}_{11}^* - \bar{c}_{11}^* \right) \bar{m}^3, \\ & - \frac{32[\bar{x}_{12}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 (\bar{x}_{12}^* + \bar{x}_{22}^*) \bar{e}_{12}^* + \bar{x}_{22}^* \bar{n}^4 \bar{e}_{12}^*]}{3ab\bar{m}\bar{n}} \bar{x}_{31}^* \bar{\ell} \end{aligned} \right\},$$

$$\bar{y}_{13}^* = \frac{ab}{8} \left\{ \begin{aligned} & 2\bar{e}_{22}^* \bar{x}_{33}^* \bar{x}_{32}^* \bar{n}^4 \left[(\bar{h}_{12}^* + \bar{h}_{21}^* + 4\bar{h}_{66}^*) \bar{m}^2 \left(\frac{4}{3h^2} \right)^2 - (\bar{c}_{21}^* + 2\bar{c}_{66}^* + \bar{c}_{12}^* + 2\bar{c}_{66}^*) \bar{m}^2 \frac{4}{3h^2} \right] \bar{n} + \\ & 2\bar{x}_{33}^* \bar{e}_{11}^* \bar{m} \bar{n}^4 \bar{x}_{32}^* + \frac{4}{3h^2} \left(\frac{8}{3h^2} \bar{h}_{22}^* - \bar{c}_{22}^* - \bar{c}_{22}^* \right) \bar{n}^3 + \left[18 \left(\frac{4}{3h^2} \right)^2 \bar{l}_{55} - 12\frac{4}{3h^2} \bar{g}_{55} + 2\bar{f}_{55} \right] \bar{n} \\ & + \bar{m}^2 \bar{n}^2 \bar{x}_{32}^* (4\bar{e}_{12}^* + 2\bar{e}_{66}^*) \bar{x}_{33}^* - \frac{32[\bar{x}_{12}^* \bar{m}^4 \bar{e}_{22}^* + \bar{e}_{12}^* \bar{m}^2 \bar{n}^2 (\bar{x}_{12}^* + \bar{x}_{22}^*) + \bar{x}_{22}^* \bar{n}^4 \bar{e}_{11}^*]}{3ab\bar{m}\bar{n}} \bar{\ell} \bar{x}_{32}^* \end{aligned} \right\},$$

$$\bar{y}_{14}^* = \frac{-8[\bar{x}_{11}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 (\bar{x}_{11}^* + \bar{x}_{21}^*) \bar{e}_{12}^* + \bar{n}^4 \bar{x}_{21}^* \bar{e}_{11}^*]}{3\bar{m}\bar{n}} \bar{\ell} \bar{x}_{31}^*,$$

$$\bar{y}_{15}^* = \frac{-8[\bar{x}_{11}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 (\bar{x}_{11}^* + \bar{x}_{21}^*) \bar{e}_{12}^* + \bar{n}^4 \bar{x}_{21}^* \bar{e}_{11}^*]}{3\bar{m}\bar{n}} \bar{\ell} \bar{x}_{32}^*,$$

$$\bar{y}_{16}^* = -\frac{4[\bar{x}_{11}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 (\bar{x}_{11}^* + \bar{x}_{21}^*) \bar{e}_{12}^* + \bar{n}^4 \bar{x}_{21}^* \bar{e}_{11}^*]}{\bar{m}\bar{n}} \bar{\ell} \bar{x}_{33}^* + 24ab[\bar{x}_{11}^* \bar{x}_{12}^* \bar{m}^4 \bar{e}_{22}^* + \bar{x}_{21}^* \bar{x}_{22}^* \bar{n}^4 \bar{e}_{11}^*],$$

$$\bar{y}_{17}^* = 16ba[\bar{x}_{11}^{*2} \bar{e}_{22}^* \bar{m}^4 + \bar{x}_{21}^{*2} \bar{e}_{11}^* \bar{n}^4], \quad \bar{y}_{18}^* = \frac{4}{3h^2} \frac{[\bar{m}^2 \bar{a}_{11}^* + \bar{n}^2 \bar{a}_{12}^*]}{\bar{m}\bar{n}} \bar{\ell}$$

$$\bar{y}_{19}^* = \bar{m}^2 ab/4, \quad \bar{y}_{110}^* = \frac{-\bar{x}_{33}^* [\bar{m}^2 \bar{e}_{22}^* + \bar{n}^2 \bar{e}_{12}^*]}{\bar{m}\bar{n}} \bar{\ell}, \quad \bar{y}_{112}^* = \frac{\bar{\ell}}{\bar{m}\bar{n}},$$

$$\bar{y}_{113}^* = -\frac{\left[\left(\bar{a}_{11}^* \Phi_{1x} + \bar{a}_{21}^* \Phi_{1y} - \Phi_{4x}\right) \bar{m}^2\right] \frac{4}{3h^2} \bar{\ell}}{\bar{m} \bar{n}} - \frac{\bar{n}^2 \left(\bar{a}_{12}^* \Phi_{1x} + \bar{a}_{22}^* \Phi_{1y} - \Phi_{4y}\right) \frac{4}{3h^2} \bar{\ell}}{\bar{m} \bar{n}},$$

$$\bar{y}_{115}^* = \frac{ab}{4} \left\{ \begin{aligned} &\mu_1 \left[\int_{-h/2}^{h/2} \rho(z) dz + \frac{b_{1y}}{d_{1y}} \int_{h/2}^{h/2+h_s} \rho dz + \frac{2b_{1l}}{d_{1l}} \int_{h/2}^{h/2+h_s} \rho dz \right] + \mu_2 \left[\int_{-h/2}^{h/2} \rho(z) dz + \frac{b_{2x}}{d_{2x}} \int_{h/2}^{h/2+h_s} \rho dz + \frac{2b_{2l}}{d_{2l}} \int_{h/2}^{h/2+h_s} \rho dz \right] \\ &+ \mu_3 \left[\int_{-h/2}^{h/2} \rho(z) dz + \frac{b_{3x}}{d_{3x}} \int_{h/2}^{h/2+h_s} \rho dz + \frac{b_{3y}}{d_{3y}} \int_{h/2}^{h/2+h_s} \rho dz + \frac{2b_{3l}}{d_{3l}} \int_{h/2}^{h/2+h_s} \rho dz \right] \end{aligned} \right\},$$

$$\bar{y}_{116}^* = ab/4,$$

$$\bar{y}_{22}^* = \frac{ab}{4} \left\{ \begin{aligned} &\left[\bar{e}_{22}^* \bar{m}^4 + \bar{n}^4 \bar{e}_{11}^* + \bar{m}^2 \bar{n}^2 (2\bar{e}_{12}^* + \bar{e}_{66}^*) \right] \bar{x}_{31}^* + \left[\left(\frac{4}{3h^2} \right)^2 \bar{h}_{11}^* + (-\bar{c}_{11}^* - \bar{c}_{11}^*) \frac{4}{3h^2} + \bar{b}_{11}^* \right] \bar{m}^2 \\ &- 6\bar{g}_{44} \frac{4}{3h^2} + \left[\left(\frac{4}{3h^2} \right)^2 \bar{h}_{66}^* + (-\bar{c}_{66}^* - \bar{c}_{66}^*) \frac{4}{3h^2} + \bar{b}_{66}^* \right] \bar{n}^2 + 9\bar{l}_{44} \left(\frac{4}{3h^2} \right)^2 + \bar{f}_{44} \end{aligned} \right\},$$

$$\bar{y}_{23}^* = \frac{ab}{4} \left\{ \begin{aligned} &\left[\bar{e}_{22}^* \bar{m}^4 + (2\bar{e}_{12}^* + \bar{e}_{66}^*) \bar{m}^2 \bar{n}^2 + \bar{e}_{11}^* \bar{n}^4 \right] \bar{x}_{32}^* \bar{x}_{31}^* \\ &+ \frac{1}{2} \bar{m} \bar{n} \left[(\bar{h}_{12}^* + \bar{h}_{21}^* + 2\bar{h}_{66}^*) \left(\frac{4}{3h^2} \right)^2 + \bar{b}_{21}^* + \bar{b}_{12}^* + 2\bar{b}_{66}^* + (-\bar{c}_{12}^* - \bar{c}_{21}^* - 2\bar{c}_{66}^* - \bar{c}_{12}^* - \bar{c}_{21}^* - 2\bar{c}_{66}^*) \frac{4}{3h^2} \right] \end{aligned} \right\},$$

$$\bar{y}_{24}^* = -\frac{4 \left[\bar{x}_{11}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 (\bar{x}_{11}^* + \bar{x}_{21}^*) \bar{e}_{12}^* + \bar{n}^4 \bar{x}_{21}^* \bar{e}_{11}^* \right] \bar{x}_{31}^* \bar{\ell}}{3\bar{m} \bar{n}}, \bar{y}_{25}^* = -\frac{\left(\bar{d}_{11}^* - \bar{a}_{11}^* \frac{4}{3h^2} \right) \bar{\ell}}{\bar{n}},$$

$$\bar{y}_{26}^* = -\frac{\left[\bar{m}^2 \bar{e}_{22}^* + \bar{n}^2 \bar{e}_{12}^* \right] \bar{x}_{31}^* \bar{\ell}}{\bar{m} \bar{n}}, \bar{y}_{27}^* = \frac{\left[(-\bar{a}_{11}^* \Phi_{1x} - \bar{a}_{21}^* \Phi_{1y} + \Phi_{4x}) \frac{4}{3h^2} + \bar{d}_{21}^* \Phi_{1y} \right] \bar{\ell}}{\bar{n}} + \frac{(\bar{d}_{11}^* \Phi_{1x} - \Phi_{2y}) \bar{\ell}}{\bar{n}},$$

$$\bar{y}_{33}^* = \frac{ab}{4} \left\{ \begin{aligned} &\left[\bar{e}_{22}^* \bar{m}^4 + \bar{n}^4 \bar{e}_{11}^* + \bar{m}^2 \bar{n}^2 (2\bar{e}_{12}^* + \bar{e}_{66}^*) \right] \bar{x}_{32}^* + \left[\left(\frac{4}{3h^2} \right)^2 \bar{h}_{22}^* + (-\bar{c}_{22}^* - \bar{c}_{22}^*) \frac{4}{3h^2} + \bar{b}_{22}^* \right] \bar{n}^2 \\ &- 6\bar{g}_{55} \frac{4}{3h^2} + \left[\left(\frac{4}{3h^2} \right)^2 \bar{h}_{66}^* + (-\bar{c}_{66}^* - \bar{c}_{66}^*) \frac{4}{3h^2} + \bar{b}_{66}^* \right] \bar{m}^2 + 9\bar{l}_{55} \left(\frac{4}{3h^2} \right)^2 + \bar{f}_{55} \end{aligned} \right\},$$

$$\bar{y}_{34}^* = -\frac{4 \left[\bar{x}_{11}^* \bar{e}_{22}^* \bar{m}^4 + \bar{m}^2 \bar{n}^2 (\bar{x}_{11}^* + \bar{x}_{21}^*) \bar{e}_{12}^* \right] \bar{x}_{32}^* \bar{\ell}}{3\bar{m} \bar{n}} + \frac{\bar{n}^4 \bar{x}_{21}^* \bar{e}_{11}^* \bar{x}_{32}^* \bar{\ell}}{3\bar{m} \bar{n}},$$

$$\bar{y}_{35}^* = \frac{\left(\bar{a}_{12}^* \frac{4}{3h^2} - \bar{d}_{12}^* \right) \bar{\ell}}{\bar{m}}, \bar{y}_{36}^* = -\frac{\bar{x}_{32}^* \left[\bar{m}^2 \bar{e}_{22}^* + \bar{n}^2 \bar{e}_{12}^* \right] \bar{\ell}}{\bar{m} \bar{n}},$$

$$\bar{y}_{37}^* = \frac{\left[(-\bar{a}_{12}^* \Phi_{1x} - \bar{a}_{22}^* \Phi_{1y} + \Phi_{4y}) \frac{4}{3h^2} + \bar{d}_{22}^* \Phi_{1y} + \bar{d}_{12}^* \Phi_{1x} - \Phi_{2y} \right] \bar{\ell}}{\bar{m}},$$

$$\bar{b}_{11}^* = -\bar{d}_{11}^* \bar{d}_{11}^* - \bar{d}_{12}^* \bar{d}_{21}^* + \bar{b}_{11}, \bar{b}_{12}^* = -\bar{d}_{11}^* \bar{d}_{12}^* - \bar{d}_{12}^* \bar{d}_{22}^* + \bar{b}_{12},$$

$$\bar{b}_{66}^* = -\bar{d}_{66}^* \bar{d}_{66}^* + \bar{b}_{66}, \bar{b}_{21}^* = -\bar{d}_{12}^* \bar{d}_{11}^* - \bar{d}_{22}^* \bar{d}_{21}^* + \bar{b}_{12}, \bar{b}_{22}^* = -\bar{d}_{12}^* \bar{d}_{12}^* - \bar{d}_{22}^* \bar{d}_{22}^* + \bar{b}_{22},$$

$$\bar{c}_{11}^* = -\bar{d}_{11}\bar{a}_{11}^* - \bar{d}_{12}\bar{a}_{21}^* + \bar{c}_{11}, \bar{c}_{12}^* = -\bar{d}_{11}\bar{a}_{12}^* - \bar{d}_{12}\bar{a}_{22}^* + \bar{c}_{12}, \bar{c}_{66}^* = -\bar{d}_{66}\bar{a}_{66}^* + \bar{c}_{66},$$

$$\bar{c}_{21}^* = -\bar{d}_{12}\bar{a}_{11}^* - \bar{d}_{22}\bar{a}_{21}^* + \bar{c}_{12}, \bar{c}_{22}^* = -\bar{d}_{12}\bar{a}_{12}^* - \bar{d}_{22}\bar{a}_{22}^* + \bar{c}_{22},$$

$$c_{11}^* = -\bar{a}_{11}\bar{d}_{11}^* - \bar{a}_{12}\bar{d}_{21}^* + \bar{c}_{11}, c_{12}^* = -\bar{a}_{11}\bar{d}_{12}^* - \bar{a}_{12}\bar{d}_{22}^* + \bar{c}_{12}, c_{66}^* = -\bar{a}_{66}\bar{d}_{66}^* + \bar{c}_{66},$$

$$c_{21}^* = -\bar{a}_{12}\bar{d}_{11}^* - \bar{a}_{22}\bar{d}_{21}^* + \bar{c}_{12}, c_{22}^* = -\bar{a}_{12}\bar{d}_{12}^* - \bar{a}_{22}\bar{d}_{22}^* + \bar{c}_{22},$$

$$\bar{h}_{11}^* = -\bar{a}_{11}\bar{a}_{11}^* - \bar{a}_{12}\bar{a}_{21}^* + \bar{h}_{11}, \bar{h}_{12}^* = -\bar{a}_{11}\bar{a}_{12}^* - \bar{a}_{12}\bar{a}_{22}^* + \bar{h}_{12}, \bar{h}_{66}^* = -\bar{a}_{66}\bar{a}_{66}^* + \bar{h}_{66},$$

$$\bar{h}_{21}^* = -\bar{a}_{12}\bar{a}_{11}^* - \bar{a}_{22}\bar{a}_{21}^* + \bar{h}_{12}, \bar{h}_{22}^* = -\bar{a}_{12}\bar{a}_{12}^* - \bar{a}_{22}\bar{a}_{22}^* + \bar{h}_{22}.$$

$$\bar{y}_{41}^* = \frac{-\left[\bar{m}^2\left(\bar{x}_{33}^*\bar{e}_{12}^* - \frac{4}{3h^2}\bar{a}_{11}^*\right) - \bar{n}^2\left(\frac{4}{3h^2}\bar{a}_{12}^* - \bar{x}_{33}^*\bar{e}_{11}^*\right)\right]\bar{\ell}}{\bar{m}\bar{n}},$$

$$\bar{y}_{42}^* = \frac{\left[\bar{x}_{31}^*\bar{m}^2\bar{e}_{12}^* + \left(\bar{d}_{11}^* - \frac{4}{3h^2}\bar{a}_{11}^*\right)\bar{m} + \bar{n}^2\bar{x}_{31}^*\bar{e}_{11}^*\right]\bar{\ell}}{\bar{m}\bar{n}}, \bar{y}_{43}^* = \frac{\left[\bar{x}_{32}^*\bar{n}^2\bar{e}_{11}^* + \left(\bar{d}_{12}^* - \frac{4}{3h^2}\bar{a}_{12}^*\right)\bar{n} + \bar{x}_{32}^*\bar{m}^2\bar{e}_{12}^*\right]\bar{\ell}}{\bar{m}\bar{n}},$$

$$\bar{y}_{44}^* = \frac{-ab\bar{m}^2}{8}, \bar{y}_{45}^* = \bar{e}_{11}^*ab, \bar{y}_{46}^* = \bar{e}_{12}^*ab, \bar{y}_{47}^* = (\bar{e}_{11}^*\Phi_{1x} + \bar{e}_{12}^*\Phi_{1y})ab,$$

$$\bar{y}_{51}^* = \frac{\left[\bar{m}^2\left(\frac{4}{3h^2}\bar{a}_{21}^* - \bar{x}_{33}^*\bar{e}_{22}^*\right) + \bar{n}^2\left(\frac{4}{3h^2}\bar{a}_{22}^* - \bar{x}_{33}^*\bar{e}_{12}^*\right)\right]\bar{\ell}}{\bar{m}\bar{n}} + \bar{R}_{4[c,p,s]},$$

$$\bar{y}_{52}^* = \frac{-\left[\bar{m}^2\bar{x}_{31}^*\bar{e}_{22}^* + \left(\frac{4}{3h^2}\bar{a}_{21}^* - \bar{d}_{21}^*\right)\bar{m} + \bar{n}^2\bar{x}_{31}^*\bar{e}_{12}^*\right]\bar{\ell}}{\bar{m}\bar{n}}, \bar{y}_{53}^* = \frac{-\left[\bar{x}_{32}^*\bar{n}^2\bar{e}_{12}^* + \left(\frac{4}{3h^2}\bar{a}_{22}^* - \bar{d}_{22}^*\right)\bar{n} + \bar{x}_{32}^*\bar{m}^2\bar{e}_{22}^*\right]\bar{\ell}}{\bar{m}\bar{n}},$$

$$\bar{y}_{53}^* = \frac{-\left[\bar{x}_{32}^*\bar{n}^2\bar{e}_{12}^* + \left(\frac{4}{3h^2}\bar{a}_{22}^* - \bar{d}_{22}^*\right)\bar{n} + \bar{x}_{32}^*\bar{m}^2\bar{e}_{22}^*\right]\bar{\ell}}{\bar{m}\bar{n}}, \bar{y}_{54}^* = -ab\bar{n}^2/8, \bar{y}_{55}^* = ba\bar{e}_{12}^*,$$

$$\bar{y}_{56}^* = ba\bar{e}_{22}^*, \bar{y}_{57}^* = ab(\bar{e}_{12}^*\Phi_{1x} + \bar{e}_{22}^*\Phi_{1y}), \bar{R}_{4[c]} = \frac{8\chi\bar{\ell}}{(4\chi^2 + b^2)\bar{m}\bar{n}}, \bar{R}_{4[p,s]} = \frac{[(-1)^m - 1]}{\bar{m}} \int_0^b \frac{\sin(\bar{n}y)}{R_{[p,s]}} dy.$$