



Article info

Type of article:

Original research paper

DOI:

<https://doi.org/10.58845/jstt.utt.2026.en.6.4.248-264>

***Corresponding author:**

Email address:

anhcc@ut.edu.vn

Received: 29/03/2026

Received in Revised Form:

22/06/2026

Accepted: 19/07/2026

Nonlinear large-deflection dynamic buckling pressure of helically stiffened FG-GRC toroidal shell segments with porous core

Nguyen Thi Phuong^{1,2}, Cao Cong Anh^{3,4,*}, Nguyen Van Tien⁵

¹Mechanics of Advanced Materials and Structures, Institute for Advanced Study in Technology, Ton Duc Thang University, Ho Chi Minh City, Vietnam

²Faculty of Civil Engineering, Ton Duc Thang University, Ho Chi Minh City, Vietnam

³University of Transport Ho Chi Minh City, Ho Chi Minh City, Vietnam

⁴Graduate University of Science and Technology, Vietnam Academy of Science and Technology, Hanoi, Vietnam

⁵Mechanics of Advanced Materials and Structures, University of Transport Technology, Hanoi, Vietnam

Abstract: This paper presents an analytical investigation of the nonlinear dynamic buckling behavior of hollow-core functionally graded graphene-reinforced composite (FG-GRC) circular toroidal shell segments subjected to rapidly increasing external pressure. The shell structure is composed of FG-GRC face sheets and a porous core, and is reinforced by orthogonal and helical stiffeners. The mechanical response of the stiffeners is modeled using anisotropic beam theory, while a coordinate transformation approach is employed to incorporate the effects of inclined stiffeners into an equivalent smeared formulation. The governing equations are established based on Donnell's shell theory in conjunction with von Kármán geometric nonlinearity, accounting for thermal effects and material gradation. By introducing an appropriate displacement field and stress function, the problem is reduced to a set of nonlinear ordinary differential equations through the Galerkin method. The dynamic response under linearly time-dependent pressure is obtained using a fourth-order Runge-Kutta scheme. The dynamic buckling load is determined according to the Budiansky-Roth criterion. Numerical results are presented to examine the influences of porous core configuration, graphene distribution patterns, stiffener arrangement, geometric parameters, loading rate, and temperature on the dynamic stability of the shell segments.

Keywords: Functionally graded graphene-reinforced composite (FG-GRC), Toroidal shell segments, Nonlinear pressure buckling, Dynamic problem, Helical stiffener.

1. Introduction

Shell structures of revolution, such as cylindrical shells, toroidal shell segments, and conical shells, have attracted considerable attention in engineering applications owing to their

high load-carrying efficiency and structural adaptability. These geometries enable effective resistance against various types of mechanical actions, including axial compression, external pressure, and torsional effects. As a result, they are

extensively employed in critical structural systems, such as pressure vessels, pipelines, aerospace components, and marine structures, where both strength and stability are of primary importance.

The mechanical responses of functionally graded material (FGM) cylindrical shells have been widely investigated using a variety of advanced theoretical and numerical approaches. Gao et al. [1] examined the nonlinear dynamic buckling of orthotropic FGM cylindrical shells subjected to longitudinal velocity, taking into account initial geometric imperfections and damping effects within the framework of an improved Donnell shell theory combined with von Kármán-type nonlinearity. Liang et al. [2] developed a semi-analytical formulation based on the differential quadrature method together with Durbin's inverse Laplace transform to study transient vibration characteristics, highlighting the significant influence of temperature variations on peak displacement responses. Ramezani et al. [3] further explored the nonlinear dynamic behavior of sandwich FGM cylindrical shells with shape memory alloy cores by employing higher-order shear deformation theory (HSDT) and semi-assumed natural strain fields, while incorporating coupled thermo-mechanical effects. In addition, Shen [4, 5] investigated the postbuckling responses of hybrid FGM cylindrical shells integrated with piezoelectric actuators, demonstrating the pronounced impact of thermo-electro-mechanical coupling on structural stability. The vibration behavior and principal parametric resonance of FGM cylindrical shells under stochastic and magneto-thermo-elastic excitations were also reported in [6, 7]. More recently, Nam et al. [8] analyzed the nonlinear response of FGM cylindrical shells reinforced with helical stiffeners under torsional loading conditions, revealing that helical stiffening configurations provide superior buckling and postbuckling performance compared to conventional orthogonal stiffeners.

In addition to conventional FGM structures,

carbon nanotube-reinforced composite (CNTRC) cylindrical shells have also been extensively studied due to their enhanced mechanical performance and multifunctional capabilities. Numerous investigations have focused on the influence of functionally graded distributions of carbon nanotubes on both stability and dynamic responses under various loading conditions. Shen [9, 10] conducted comprehensive analyses on the postbuckling behavior of FG-CNTRC cylindrical shells subjected to axial compression and external pressure, incorporating the effects of thermal environments and large-deflection nonlinearities. Asadi and Wang [11] examined aeroelastic buckling and flutter characteristics of FG-CNTRC cylindrical shells in supersonic airflow by employing the first-order shear deformation theory (FSDT) in conjunction with the quasi-steady Krumhaar modified piston theory. Using the same theoretical framework, Sun et al. [12] investigated nonlinear forced vibrations through the application of the harmonic balance method. Meanwhile, the free vibration characteristics of skew FG-CNTRC cylindrical shells were analyzed by adopting a Chebyshev-Ritz approach within an oblique coordinate system [13]. Furthermore, Ansari et al. [14] explored the vibration behavior of sandwich FG-CNTRC cylindrical shells under internal pressure using the variational differential quadrature method combined with higher-order shear deformation theory. The role of stiffening elements was also addressed by Dong et al. [15], who demonstrated that the incorporation of FG-CNTRC stiffeners, modeled via an improved anisotropic smeared technique, can significantly enhance the nonlinear buckling resistance of cylindrical shells.

Functionally graded graphene-reinforced composite (FG-GRC) cylindrical shells and toroidal shell segments have been widely investigated due to their superior stiffness, strength, and thermal resistance compared to conventional composites. Considerable efforts have been devoted to

understanding their stability and dynamic characteristics under various loading conditions. Shen and Xiang [16–18] examined the static and postbuckling behavior of FG-GRC cylindrical shells subjected to combined mechanical and thermal loads, demonstrating that appropriately tailored graphene distributions can significantly enhance structural stability, whereas elevated temperatures tend to reduce critical load levels. More generally, Li et al. [19] further highlighted the importance of imperfection-sensitive modeling in postbuckling predictions for anisotropic laminated cylindrical shells. For stiffened configurations, Phuong et al. [20] proposed an improved smeared stiffener formulation that incorporates anisotropic effects, showing that FG-GRC stiffeners markedly increase buckling resistance, particularly in thermomechanical environments. In terms of dynamic behavior, Shen et al. [21] investigated nonlinear vibration responses and reported that both temperature variations and graphene grading patterns play key roles in governing dynamic stability. Lei and Tong [22] also analyzed low-velocity impact responses, emphasizing the influence of material gradation on transient structural performance. With regard to toroidal shell segments, Dong et al. [23] employed the Ritz method to study the nonlinear torsional buckling of sandwich shells with FG-GRC face sheets, revealing the beneficial effects of corrugated cores on structural stability. Similarly, Phuong et al. [24] examined the nonlinear buckling of auxetic-core toroidal shell segments under radial loading, demonstrating that auxetic cores combined with FG-GRC layers can substantially improve postbuckling strength. Related investigations on conical shells and plate structures were conducted by Kiani [25] and Abbaspour et al. [26] using the first-order shear deformation theory. In addition, functionally graded graphene platelet reinforced composites (FG-GPLRC) have been explored for both cylindrical and toroidal shells, further enriching the understanding of graphene-based

composite structures. More recently, the nonlinear vibration and dynamic buckling of stiffened FG-GRC panels with complex curvatures were analyzed using the energy method in combination with an approximate stress function approach [27]. The static buckling and postbuckling problems of FG-GPLRC and FG-CNTRC cylindrical shells and toroidal shell segments stiffened by FG-GPLRC and FG-CNTRC stiffeners, respectively, subjected to external pressure and axial compression were also studied [28–31].

Despite extensive studies on FGM, CNTRC, and FG-GRC shell structures, limited attention has been paid to the nonlinear dynamic buckling under transient loading, particularly for shells subjected to rapidly increasing external pressure. Moreover, the coupled effects of porous core configurations and graphene gradation on dynamic stability remain insufficiently understood. Existing works on stiffened shells have also mainly considered orthogonal or helical stiffeners separately, while their combined influence has not been adequately addressed. In addition, studies on the dynamic stability of FG-GRC toroidal shell segments are still scarce.

Motivated by these limitations, this study develops an analytical model for the nonlinear dynamic buckling of hollow-core FG-GRC circular toroidal shell segments reinforced by combined orthogonal and helical stiffeners under rapidly increasing external pressure. The formulation is based on Donnell's shell theory with von Kármán nonlinearity, incorporating material gradation, porous core effects, and thermal conditions. The governing equations are derived using an energy approach and reduced via the Galerkin method to a system of nonlinear time-dependent equations, which are solved by a fourth-order Runge-Kutta scheme. The dynamic buckling load is determined using the Budiansky-Roth criterion, and parametric studies are performed to evaluate the effects of structural and material parameters on dynamic stability.

The inclination angle χ of the helical stiffeners with respect to the meridional axis Ox is related to the number of helical stiffeners n_{sp} and their spacing d_{sp} . This relationship can be expressed as $\chi = \arccos\left(\frac{n_{sp}d_{sp}}{2\pi R}\right)$

The graphene sheets exhibit two principal crystallographic directions, namely the zigzag and armchair orientations. In the present study, the zigzag direction is aligned with the longitudinal direction of the shell, corresponding to the 0° layer, while the armchair direction is associated with the circumferential direction, corresponding to the 90° layer. The face sheets and the stiffeners are composed of multiple graphene-reinforced composite (GRC) layers, and for simplicity, a representative stacking sequence of $(0)_{10}$ is adopted.

Three types of graphene distribution are considered, including uniform distribution (UD) and two functionally graded patterns [16-18], namely FG-X and FG-O. For the FG-O configuration, the graphene volume fraction varies symmetrically about the mid-plane according to the sequence $[0.11/0.09/0.07/0.05/0.03]_s$, whereas the FG-X pattern follows the reversed sequence $[0.03/0.05/0.07/0.09/0.11]_s$. In contrast, the UD distribution assumes a constant graphene volume fraction across all layers ($V_G = 0.07$).

The core layer is assumed to consist of the same material as the matrix phase. The effective mechanical and thermal properties of the porous core, namely Young's modulus, thermal expansion coefficient, and Poisson's ratio, are determined by considering the influence of porosity, estimated as

$$\begin{cases} E_{co} = E_M \left[1 - e_1 \cos\left(\frac{\pi z}{h_{co}}\right) \right], & -\frac{h_{co}}{2} < z < \frac{h_{co}}{2}, \\ \alpha_{co} = \alpha_M \\ \rho_{co} = \rho_M \left[1 - \left(1 - \sqrt{1 - e_1}\right) \cos\left(\frac{\pi z}{h_{co}}\right) \right], \end{cases} \quad (1)$$

where e_1 ($0 \leq e_1 \leq 1$) is the porosity coefficient.

By combining the GRC face sheets, porous core layer, and GRC stiffeners, three structural configurations are established, denoted as UD-PC-UD/UD, X-PC-X/X, and O-PC-O/O. In this paper, the material parameters of graphene and the matrix are referred to the researches of Shen and Xiang [16-18].

3. Governing equations

In this study, the Donnell shell theory combined with von Kármán geometric nonlinearity is employed to establish the governing equations for the nonlinear dynamic stability analysis of FG-GRC toroidal shell segments subjected to external pressure. The strain components through the shell thickness at a distance z from the mid-surface can be expressed as follows [23, 24]

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \hat{\varepsilon}_x^0 \\ \hat{\varepsilon}_y^0 \\ \hat{\gamma}_{xy}^0 \end{bmatrix} - z \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}, \quad (2)$$

where $\hat{\varepsilon}_x^0$ and $\hat{\varepsilon}_y^0$ denote the in-plane normal strains, and $\hat{\gamma}_{xy}^0$ represents the in-plane shear strain at the mid-surface of the shell, and

$$\begin{bmatrix} \hat{\varepsilon}_x^0 \\ \hat{\varepsilon}_y^0 \\ \hat{\gamma}_{xy}^0 \end{bmatrix} = \begin{bmatrix} u_{,x} - w/a + w_{,x}^2/2 \\ -w/R + v_{,y} + w_{,y}^2/2 \\ v_{,x} + w_{,x}w_{,y} + u_{,y} \end{bmatrix}, \quad (3)$$

where $u = u(x, y)$, $v = v(x, y)$, and $w = w(x, y)$ are the displacement components along the x , y and z -directions, respectively.

The strain compatibility equation of the shell can be derived from Eq. (3) and is expressed as follows

$$\begin{aligned} \hat{\varepsilon}_{y,xx}^0 - \hat{\gamma}_{xy,xy}^0 + \hat{\varepsilon}_{x,yy}^0 = \\ w_{,xy}^2 - w_{,xx}w_{,yy} - \frac{1}{a}w_{,yy} - \frac{1}{R}w_{,xx}. \end{aligned} \quad (4)$$

The constitutive relations for FG-GRC toroidal shell segments are established based on Hooke's law and can be written as follows

$$\begin{aligned}\sigma_x &= \hat{Q}_{11}\varepsilon_x + \hat{Q}_{12}\varepsilon_y - \alpha_{11}\Delta T, \\ \sigma_y &= \hat{Q}_{12}\varepsilon_x + \hat{Q}_{22}\varepsilon_y - \alpha_{22}\Delta T, \\ \sigma_{xy} &= \hat{Q}_{66}\gamma_{xy},\end{aligned}\quad (5)$$

where the reduced stiffnesses for the shell skin

$$\begin{aligned}\hat{Q}_{11} &= \frac{E_{11}}{1 - \nu_{12}\nu_{21}}, \quad \hat{Q}_{12} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}}, \\ \hat{Q}_{22} &= \frac{E_{22}}{1 - \nu_{12}\nu_{21}}, \quad \hat{Q}_{66} = G_{12},\end{aligned}$$

and for the porous core

$$\hat{Q}_{11} = \hat{Q}_{22} = \frac{E_{co}}{1 - \nu_M^2}, \quad \hat{Q}_{12} = \frac{\nu_M E_{co}}{1 - \nu_M^2}, \quad \hat{Q}_{66} = \frac{E_{co}}{2(1 + \nu_M)},$$

The FG-GRC stiffeners are modeled as anisotropic beams, and their mechanical contributions are incorporated into the shell behavior by employing the smeared stiffener technique. Furthermore, a coordinate transformation technique is applied to account for the helical stiffeners, allowing their forces and moments to be consistently integrated into the global force and moment resultants of the shell, as follows

$$\begin{aligned}\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} &= \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & 0 & \hat{B}_{11} & \hat{B}_{12} & 0 \\ \hat{A}_{12} & \hat{A}_{22} & 0 & \hat{B}_{12} & \hat{B}_{22} & 0 \\ 0 & 0 & \hat{A}_{66} & 0 & 0 & \hat{B}_{66} \\ \hat{B}_{11} & \hat{B}_{12} & 0 & \hat{D}_{11} & \hat{D}_{12} & 0 \\ \hat{B}_{12} & \hat{B}_{22} & 0 & \hat{D}_{12} & \hat{D}_{22} & 0 \\ 0 & 0 & \hat{B}_{66} & 0 & 0 & \hat{D}_{66} \end{bmatrix} \times \\ &\begin{bmatrix} \hat{\varepsilon}_x^0 \\ \hat{\varepsilon}_y^0 \\ \hat{\gamma}_{xy}^0 \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{bmatrix} - \begin{bmatrix} \hat{\Phi}_{1x} \\ \hat{\Phi}_{1y} \\ 0 \\ \hat{\Phi}_{2x} \\ \hat{\Phi}_{2y} \\ 0 \end{bmatrix} \Delta T,\end{aligned}\quad (6)$$

where

$$\begin{aligned}\hat{A}_{11} &= \hat{A}_{11}^{sh} + \bar{\mu}_1 \hat{A}_{11}^{st} + 2\bar{\mu}_2 \hat{A}_{11}^{sp} c^4, \\ \hat{A}_{22} &= \hat{A}_{22}^{sh} + \bar{\mu}_1 \hat{A}_{22}^{ri} + 2\bar{\mu}_2 \hat{A}_{11}^{sp} s^4, \\ \hat{A}_{12} &= \hat{A}_{12}^{sh} + 2\bar{\mu}_2 \hat{A}_{11}^{sp} s^2 c^2, \\ \hat{A}_{66} &= \hat{A}_{66}^{sh} + 2\bar{\mu}_2 \hat{A}_{11}^{sp} s^2 c^2,\end{aligned}$$

$$\begin{aligned}\hat{B}_{11} &= \hat{B}_{11}^{sh} + \bar{\mu}_1 \hat{B}_{11}^{st} + 2\bar{\mu}_2 \hat{B}_{11}^{sp} c^4, \\ \hat{B}_{22} &= \hat{B}_{22}^{sh} + \bar{\mu}_1 \hat{B}_{22}^{ri} + 2\bar{\mu}_2 \hat{B}_{11}^{sp} s^4, \\ \hat{B}_{12} &= \hat{B}_{12}^{sh} + 2\bar{\mu}_2 \hat{B}_{11}^{sp} s^2 c^2, \\ \hat{B}_{66} &= \hat{B}_{66}^{sh} + 2\bar{\mu}_2 \hat{B}_{11}^{sp} s^2 c^2, \\ \hat{D}_{11} &= \hat{D}_{11}^{sh} + \bar{\mu}_1 \hat{D}_{11}^{st} + 2\bar{\mu}_2 \hat{D}_{11}^{sp} c^4, \\ \hat{D}_{22} &= \hat{D}_{22}^{sh} + \bar{\mu}_1 \hat{D}_{22}^{ri} + 2\bar{\mu}_2 \hat{D}_{11}^{sp} s^4, \\ \hat{D}_{12} &= \hat{D}_{12}^{sh} + 2\bar{\mu}_2 \hat{D}_{11}^{sp} s^2 c^2, \\ \hat{D}_{66} &= \hat{D}_{66}^{sh} + 2\bar{\mu}_2 \hat{D}_{11}^{sp} s^2 c^2, \\ \hat{\Phi}_{1x} &= \hat{\Phi}_{1x}^{sh} + \bar{\mu}_1 \hat{\Phi}_{1x}^{st} + 2\bar{\mu}_2 \hat{\Phi}_{1x}^{sp} (c^6 + s^2 c^2), \\ \hat{\Phi}_{2x} &= \hat{\Phi}_{2x}^{sh} + \bar{\mu}_1 \hat{\Phi}_{2x}^{st} + 2\bar{\mu}_2 \hat{\Phi}_{2x}^{sp} (c^6 + s^2 c^2), \\ \hat{\Phi}_{1y} &= \hat{\Phi}_{1y}^{sh} + \bar{\mu}_1 \hat{\Phi}_{1y}^{ri} + 2\bar{\mu}_2 \hat{\Phi}_{1x}^{sp} (c^6 + s^2 c^2), \\ \hat{\Phi}_{2y} &= \hat{\Phi}_{2y}^{sh} + \bar{\mu}_1 \hat{\Phi}_{2y}^{ri} + 2\bar{\mu}_2 \hat{\Phi}_{2x}^{sp} (c^6 + s^2 c^2),\end{aligned}$$

$$s = \sin \chi, \quad c = \cos \chi,$$

where $\bar{\mu}_1$ and $\bar{\mu}_2$ are indicator coefficients used to characterize the presence of stiffeners. The case $\bar{\mu}_1 = 1, \bar{\mu}_2 = 0$ corresponds to orthogonal stiffeners, $\bar{\mu}_1 = 0, \bar{\mu}_2 = 1$ represents helical stiffeners, whereas $\bar{\mu}_1 = 0, \bar{\mu}_2 = 0$ denotes the unstiffened configuration, and

$$(\hat{A}_{ij}^{sh}, \hat{B}_{ij}^{sh}, \hat{D}_{ij}^{sh}) = \int_{-h/2}^{h/2} \hat{Q}_{ij}(z) (1, z, z^2) dz, \quad (i, j = 1, 2, 6),$$

$$(\hat{\Phi}_{1x}^{sh}, \hat{\Phi}_{2x}^{sh}) = \int_{-h/2}^{h/2} \begin{bmatrix} \hat{Q}_{11}(z) \alpha_{11}(z) + \\ \hat{Q}_{12}(z) \alpha_{22}(z) \end{bmatrix} (1, z^2) dz,$$

$$(\hat{\Phi}_{1y}^{sh}, \hat{\Phi}_{2y}^{sh}) = \int_{-h/2}^{h/2} \begin{bmatrix} \hat{Q}_{12}(z) \alpha_{11}(z) + \\ \hat{Q}_{22}(z) \alpha_{22}(z) \end{bmatrix} (1, z^2) dz,$$

$$(\hat{\Phi}_{1x}^{st}, \hat{\Phi}_{2x}^{st}) = (\hat{\Phi}_{1x}^{sp}, \hat{\Phi}_{2x}^{sp}) =$$

$$\frac{b_{(st,sp)}}{d_{(st,sp)}} \int_{h/2}^{h/2+h_{(st,sp)}} \begin{bmatrix} \hat{Q}_{11}(z) \alpha_{11}(z) + \\ \hat{Q}_{12}(z) \alpha_{22}(z) \end{bmatrix} (1, z^2) dz,$$

$$(\hat{\Phi}_{1y}^{ri}, \hat{\Phi}_{2y}^{ri}) = \frac{b_{ri}}{d_{ri}} \int_{h/2}^{h/2+h_{ri}} \begin{bmatrix} \hat{Q}_{12}(z) \alpha_{11}(z) + \\ \hat{Q}_{22}(z) \alpha_{22}(z) \end{bmatrix} (1, z^2) dz,$$

The equivalent stiffness coefficients of the meridional and inclined stiffeners, namely $\hat{A}_{11}^{(st,sp)}, \hat{B}_{11}^{(st,sp)}, \hat{D}_{11}^{(st,sp)}$, are obtained as follows:

$$\begin{bmatrix} \hat{A}_{11}^{(st,sp)} & \hat{B}_{11}^{(st,sp)} \\ \hat{B}_{11}^{(st,sp)} & \hat{D}_{11}^{(st,sp)} \end{bmatrix} = \begin{bmatrix} A_{11}^* & B_{11}^* \\ B_{11}^* & D_{11}^* \end{bmatrix} - \begin{bmatrix} A_{12}^* & 0 & B_{12}^* & 0 \\ B_{12}^* & 0 & D_{12}^* & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} A_{22}^* & 0 & B_{22}^* & 0 \\ 0 & A_{66}^* & 0 & B_{66}^* \\ B_{22}^* & 0 & D_{22}^* & 0 \\ 0 & B_{66}^* & 0 & D_{66}^* \end{bmatrix}^{-1} \begin{bmatrix} A_{12}^* & B_{12}^* \\ 0 & 0 \\ B_{12}^* & D_{66}^* \\ 0 & 0 \end{bmatrix}, \quad (7)$$

where

$$(A_{ij}^*, B_{ij}^*, D_{ij}^*) = \frac{b_{(st,sp)}}{d_{(st,sp)}} \int_{h/2}^{h/2+h_{(st,sp)}} \hat{Q}_{ij}(1, z, z^2) dz, \quad (8)$$

$(i, j = 1, 2, 6)$

It is worth noting that the stiffness coefficients of the circumferential stiffeners, $\hat{A}_{22}^{ri}$, $\hat{B}_{22}^{ri}$, and $\hat{D}_{22}^{ri}$, are obtained by exchanging the indices “11” and “22”. Furthermore, the parameters $b_{(st,sp)}$ and $d_{(st,sp)}$ are substituted by b_{ri} and d_{ri} , respectively.

The Airy stress function $\hat{\phi}(x, y)$ can be introduced to satisfy the following equilibrium conditions

$$N_x = \hat{\phi}_{,yy}, \quad N_y = \hat{\phi}_{,xx}, \quad N_{xy} = -\hat{\phi}_{,xy}. \quad (9)$$

Substituting Eq. (9) into Eq. (4) yields the following expression

$$\begin{aligned} & \hat{A}_{22}^* \hat{\phi}_{,xxxx} + (\hat{A}_{66}^* + 2\hat{A}_{12}^*) \hat{\phi}_{,xxyy} + \hat{A}_{11}^* \hat{\phi}_{,yyyy} \\ & + \hat{B}_{21}^* w_{,xxxx} + (\hat{B}_{11}^* + \hat{B}_{22}^* - 2\hat{B}_{66}^*) w_{,xxyy} \\ & + \hat{B}_{12}^* w_{,yyyy} - w_{,xy}^2 + w_{,xx} w_{,yy} + \frac{1}{R} w_{,xx} + \frac{1}{a} w_{,yy} = 0. \end{aligned} \quad (10)$$

where

$$\begin{aligned} \hat{A}_{11}^* &= \frac{\hat{A}_{22}^*}{\hat{A}_{11}^* \hat{A}_{22}^* - \hat{A}_{12}^*{}^2}, \hat{A}_{22}^* = \frac{\hat{A}_{11}^*}{\hat{A}_{11}^* \hat{A}_{22}^* - \hat{A}_{12}^*{}^2}, \\ \hat{A}_{12}^* &= -\frac{\hat{A}_{12}^*}{\hat{A}_{11}^* \hat{A}_{22}^* - \hat{A}_{12}^*{}^2}, \hat{A}_{66}^* = \frac{1}{\hat{A}_{66}^*}, \\ \hat{B}_{11}^* &= \hat{A}_{11}^* \hat{B}_{11}^* + \hat{A}_{12}^* \hat{B}_{12}^*, \\ \hat{B}_{22}^* &= \hat{A}_{22}^* \hat{B}_{22}^* + \hat{A}_{12}^* \hat{B}_{12}^*, \\ \hat{B}_{12}^* &= \hat{A}_{11}^* \hat{B}_{12}^* + \hat{A}_{12}^* \hat{B}_{22}^*, \\ \hat{B}_{21}^* &= \hat{A}_{22}^* \hat{B}_{12}^* + \hat{A}_{12}^* \hat{B}_{11}^*, \quad \hat{B}_{66}^* = \frac{\hat{B}_{66}^*}{\hat{A}_{66}^*} \end{aligned}$$

4. Assumed deflection form and solution procedure

Consider FG-GRC toroidal shell segments subjected to external pressure with simply

supported boundary conditions at both ends, presented as

$$w = N_x = N_{xy} = M_x = 0 \quad \text{at } x = 0, x = L. \quad (11)$$

An approximate analytical form of the deflection is assumed as follows [24]

$$w = W_0 + W_1 \sin a_m x \sin b_n y + W_2 \sin^2 a_m x \quad (12)$$

where $a_m = \frac{m\pi}{L}$ and $b_n = \frac{n}{R}$, and it can be seen

that the deflection form satisfies the boundary conditions (11) in the average sense.

Substituting the assumed deflection function (12) into Eq. (10), the corresponding stress function is obtained as follows

$$\begin{aligned} \hat{\phi} &= \hat{\phi}_1 \cos 2a_m x + \hat{\phi}_2 \cos 2b_n y + \\ &+ \hat{\phi}_3 \sin a_m x \sin b_n y + \\ &+ \hat{\phi}_4 \sin 3a_m x \sin b_n y - \sigma_{0y} h \frac{x^2}{2}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} \hat{\phi}_1 &= k_1 W_2 + k_2 W_1^2, \quad \hat{\phi}_2 = k_3 W_1^2, \\ \hat{\phi}_3 &= k_4 W_1 W_2 + k_5 W_1, \quad \hat{\phi}_4 = k_6 W_1 W_2, \\ k_1 &= \frac{4Ra_m^2 \hat{B}_{21}^* - 1}{8Ra_m^2 \hat{A}_{22}^*}, \quad k_2 = \frac{b_n^2}{32a_m^2 \hat{A}_{22}^*}, \quad k_3 = \frac{a_m^2}{32b_n^2 \hat{A}_{11}^*}, \\ k_4 &= -\frac{b_n^2 a_m^2}{a_m^4 \hat{A}_{22}^* + a_m^2 b_n^2 (\hat{A}_{66}^* + 2\hat{A}_{12}^*) + b_n^4 \hat{A}_{11}^*}, \\ k_5 &= \frac{b_n^2 \hat{B}_{12}^* - \frac{b_n^2}{a} - \frac{a_m^2}{R}}{a_m^4 \hat{A}_{22}^* + a_m^2 b_n^2 (\hat{A}_{66}^* + 2\hat{A}_{12}^*) + b_n^4 \hat{A}_{11}^*} \\ &\quad - \frac{a_m^4 \hat{B}_{21}^* + a_m^2 b_n^2 (\hat{B}_{11}^* + \hat{B}_{22}^* - 2\hat{B}_{66}^*)}{a_m^4 \hat{A}_{22}^* + a_m^2 b_n^2 (\hat{A}_{66}^* + 2\hat{A}_{12}^*) + b_n^4 \hat{A}_{11}^*}, \\ k_6 &= \frac{b_n^2 a_m^2}{81a_m^4 \hat{A}_{22}^* + 9b_n^2 a_m^2 (\hat{A}_{66}^* + 2\hat{A}_{12}^*) + b_n^4 \hat{A}_{11}^*}. \end{aligned}$$

Furthermore, the closure condition for the toroidal shell segments is formulated in an averaged form as follows [23, 24]

$$\int_0^{2\pi R} \int_0^L v_{,y} dx dy = \int_0^{2\pi R} \int_0^L \left(\hat{\varepsilon}_y^0 + \frac{w}{R} - \frac{w_{,y}}{2} \right) dx dy = 0 \quad (14)$$

The circumferential stress σ_{0y} can be derived from Eq. (14) after some algebraic

simplifications, and is expressed as

$$\sigma_{0y} = \frac{W_2}{2Rh\hat{A}_{22}^*} - \frac{b_n^2 W_1^2}{8h\hat{A}_{22}^*} + \frac{W_0}{Rh\hat{A}_{22}^*} + \frac{(\hat{\Phi}_{1x}\hat{A}_{12}^* + \hat{\Phi}_{1y}\hat{A}_{22}^*)\Delta T}{h\hat{A}_{22}^*} \quad (15)$$

The Lagrangian of the system, including the kinetic and potential energies, can be expressed as follows

$$\Delta_{\text{Total}} = \Delta_t - \Delta_{\text{in}} + \Delta_{\text{ext}}. \quad (16)$$

The elastic strain energy stored in the shell can be expressed as

$$\Delta_{\text{in}} = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^{2\pi R} \int_0^L \left(\sigma_{xy} \gamma_{xy} + \sigma_x \varepsilon_x + \sigma_y \varepsilon_y - \sigma_x \alpha_{11} \Delta T - \sigma_y \alpha_{22} \Delta T \right) dx dy dz. \quad (17)$$

The work of the external pressure load acting on the shell is given by

$$\Delta_{\text{ext}} = \int_0^{2\pi R} \int_0^L q w dx dy. \quad (18)$$

The kinetic energy of the shell structure can be expressed as

$$\Delta_t = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^{2\pi R} \int_0^L \rho(z) w_{,t}^2 dx dy dz \quad (19)$$

The damping potential function is defined based on the Rayleigh dissipation function as

$$d_c = \frac{1}{2} \int_0^{2\pi R} \int_0^L c w_{,t}^2 dx dy. \quad (20)$$

The governing equations of motion are obtained by applying the Euler-Lagrange formulation, as follows

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \Delta_{\text{Total}}}{\partial \dot{W}_0} \right) - \frac{\partial \Delta_{\text{Total}}}{\partial W_0} + \frac{\partial d_c}{\partial \dot{W}_0} &= 0, \\ \frac{d}{dt} \left(\frac{\partial \Delta_{\text{Total}}}{\partial \dot{W}_1} \right) - \frac{\partial \Delta_{\text{Total}}}{\partial W_1} + \frac{\partial d_c}{\partial \dot{W}_1} &= 0, \\ \frac{d}{dt} \left(\frac{\partial \Delta_{\text{Total}}}{\partial \dot{W}_2} \right) - \frac{\partial \Delta_{\text{Total}}}{\partial W_2} + \frac{\partial d_c}{\partial \dot{W}_2} &= 0. \end{aligned} \quad (21)$$

By incorporating Eq. (16), Eq. (21) is transformed into the following form

$$H_{11} W_0 + H_{12} W_1^2 + H_{13} W_2 - 2q + H_{15} \Delta T + c(2\dot{W}_0 + \dot{W}_2) + I_1(2\ddot{W}_0 + \ddot{W}_2) = 0 \quad (22)$$

$$H_{21} W_0 W_1 + H_{22} W_1^3 + H_{23} W_1 W_2 + H_{24} W_1 W_2^2 + H_{25} W_1 + H_{27} W_1 \Delta T + \frac{c\dot{W}_1}{2} + \frac{I_1 \ddot{W}_1}{2} = 0 \quad (23)$$

$$H_{31} W_0 + H_{32} W_2 W_1^2 + H_{33} W_1^2 + H_{34} W_2 - q + H_{37} \Delta T + \frac{c(8\dot{W}_0 + 6\dot{W}_2)}{8} + \frac{I_1(8\ddot{W}_0 + 6\ddot{W}_2)}{8} = 0 \quad (24)$$

where

$$H_{11} = \frac{2}{R^2 \hat{A}_{22}^*}, H_{12} = -\frac{b_n^2}{4R \hat{A}_{22}^*},$$

$$H_{13} = \frac{1}{R^2 \hat{A}_{22}^*},$$

$$H_{15} = \frac{2\hat{\Phi}_{1y}\hat{A}_{22}^* + 2\hat{A}_{12}^*\hat{\Phi}_{1x}}{R\hat{A}_{22}^*},$$

$$H_{21} = -\frac{b_n^2}{2R\hat{A}_{22}^*},$$

$$H_{22} = \frac{64k_2^2 a_m^4 \hat{A}_{22}^{*2} + 64\hat{A}_{11}^* k_3^2 b_n^4 \hat{A}_{22}^* + b_n^4/8}{2\hat{A}_{22}^*},$$

$$H_{23} = (32k_2 k_1 + k_5 k_4) a_m^4 \hat{A}_{22}^* + k_5 k_4 \hat{A}_{11}^* b_n^4 + (2\hat{A}_{12}^* + \hat{A}_{66}^*) k_5 k_4 a_m^2 b_n^2 - b_n^2 / (4R \hat{A}_{22}^*),$$

$$H_{24} = \frac{1}{2} \left[(81k_6^2 + k_4^2) a_m^4 \hat{A}_{22}^* + \hat{A}_{11}^* (k_4^2 + k_6^2) b_n^4 + (2\hat{A}_{12}^* + \hat{A}_{66}^*) (9k_6^2 + k_4^2) a_m^2 b_n^2 \right],$$

$$H_{25} = \frac{1}{2} \left\{ \left(\hat{A}_{22}^* k_5^2 - \hat{F}_{11}^* \right) a_m^4 + \left(\hat{A}_{11}^* k_5^2 - \hat{F}_{22}^* \right) b_n^4 + \left[(2\hat{A}_{12}^* + \hat{A}_{66}^*) k_5^2 - \hat{F}_{12}^* - \hat{F}_{21}^* - 4\hat{F}_{66}^* \right] a_m^2 b_n^2 \right\},$$

$$H_{27} = -\frac{(\hat{\Phi}_{1y}\hat{A}_{22}^* + \hat{A}_{12}^*\hat{\Phi}_{1x})b_n^2}{2\hat{A}_{22}^*}, H_{31} = \frac{1}{R^2 \hat{A}_{22}^*},$$

$$H_{32} = \frac{1}{2} \left[a_m^4 (k_4^2 + 81k_6^2) \hat{A}_{22}^* + (2\hat{A}_{12}^* + \hat{A}_{66}^*) \times (9k_6^2 + k_4^2) a_m^2 b_n^2 + \hat{A}_{11}^* (k_4^2 + k_6^2) b_n^4 \right],$$

$$H_{33} = a_m^4 \left(16k_2 k_1 + \frac{1}{2} k_5 k_4 \right) \hat{A}_{22}^* + \frac{1}{2} \left[(2\hat{A}_{12}^* + \hat{A}_{66}^*) k_5 k_4 a_m^2 b_n^2 \right] - \frac{b_n^2}{8R \hat{A}_{22}^*},$$

$$H_{34} = 16k_1^2 a_m^4 \hat{A}_{22}^* - 4a_m^4 \hat{F}_{11}^* + \frac{1}{2R^2 \hat{A}_{22}^*},$$

$$H_{37} = \frac{\hat{\Phi}_{1y}\hat{A}_{22}^* + \hat{\Phi}_{1x}\hat{A}_{12}^*}{R\hat{A}_{22}^*},$$

$$\begin{aligned}
I_1 &= \int_{-h/2}^{h/2} \rho(z) dz + \bar{\mu}_1 \frac{b_{st}}{d_{st}} \int_{h/2}^{h/2+h_{st}} \rho(z) dz \\
&+ \bar{\mu}_1 \frac{b_{ri}}{d_{ri}} \int_{h/2}^{h/2+h_{ri}} \rho(z) dz + 2\bar{\mu}_2 \frac{b_{sp}}{d_{sp}} \int_{h/2}^{h/2+h_{sp}} \rho(z) dz, \\
\hat{F}_{11}^* &= \hat{B}_{11}^* \hat{B}_{11} + \hat{B}_{21}^* \hat{B}_{12} - \hat{D}_{11}, \\
\hat{F}_{12}^* &= \hat{B}_{12}^* \hat{B}_{11} + \hat{B}_{22}^* \hat{B}_{12} - \hat{D}_{12}, \\
\hat{F}_{21}^* &= \hat{B}_{11}^* \hat{B}_{12} + \hat{B}_{21}^* \hat{B}_{22} - \hat{D}_{12}, \\
\hat{F}_{22}^* &= \hat{B}_{12}^* \hat{B}_{12} + \hat{B}_{22}^* \hat{B}_{22} - \hat{D}_{22}, \\
\hat{F}_{66}^* &= \hat{B}_{66}^* \hat{B}_{66} - \hat{D}_{66}.
\end{aligned}$$

FG-GRC toroidal shell segments are subjected to an external pressure increasing linearly with time, expressed as $q = q_0 t$, where q_0 (Pa/s) represents the loading rate. The governing equations (22-24) are solved numerically using a Runge-Kutta scheme. The total deflection is taken as $W_{max} = W_0 + W_1 + W_2$, and the dynamic buckling load is determined based on the Budiansky-Roth criterion, which defines dynamic instability as a sudden qualitative change in the total deflection under gradually increasing load.

The upper static buckling load is determined by neglecting the nonlinear and inertial terms in Eqs. (21-24). Solving the resulting simplified system yields

$$q_{upper} = -\frac{T_{28} + T_{210} \Delta T}{T_{27}}. \quad (25)$$

Table 1. Comparison of static buckling loads q_{upper} (in kN) of FG-GRC cylindrical shells with available results in the literature ((0)₁₀, $R/h = 20$, $h = 2$ mm, $T = 300$ K).

L^2/Rh	Shen and Xiang [16]		Present	
	UD	FG-X	UD	FG-X
100	20.231 (1;4)*	22.162 (1;4)	20.885 (1;4)	23.087 (1;4)
300	11.173 (1;3)	12.082 (1;3)	11.335 (1;3)	12.346 (1;3)
500	9.525 (1;3)	10.426 (1;3)	9.678 (1;3)	10.671 (1;3)

* The buckling modes (m;n).

Table 2 compares the static and dynamic buckling loads of hollow-core FG-GRC cylindrical shells and toroidal shell segments with helical stiffeners under different temperatures and loading rates. It is clearly observed that the shell geometry

where

$$\begin{aligned}
T_{27} &= \frac{H_{31}(-H_{12}T_{15} + 2)}{H_{11}} + T_{15}H_{33} - 1, \\
T_{28} &= -\frac{H_{31}H_{12}T_{13}}{H_{11}} + T_{13}H_{33}, \\
T_{210} &= \frac{H_{31}(-H_{12}T_{16} - H_{15})}{H_{11}} + T_{16}H_{33} + H_{37}, \\
T_{15} &= -\frac{2H_{21}}{H_{11}H_{22} - H_{12}H_{21}}, \\
T_{13} &= -\frac{H_{25}H_{11}}{H_{11}H_{22} - H_{12}H_{21}}, \\
T_{16} &= -\frac{H_{11}H_{27} - H_{15}H_{21}}{H_{11}H_{22} - H_{12}H_{21}}.
\end{aligned}$$

Considering all possible buckling modes, the static critical load is obtained by selecting the minimum value among all upper-bound buckling loads.

5. Numerical results and discussion

Table 1 presents a comparison between the present results and available data in the literature for the static buckling loads of FG-GRC cylindrical shells. It can be observed that the results obtained in this study are in very good agreement with those reported by previous authors. The discrepancies between the present predictions and the reference results are relatively small, indicating the high accuracy and reliability of the proposed analytical model.

has a pronounced influence on buckling resistance. Among the considered configurations, the convex toroidal shell segment ($a = 4R$) exhibits the highest buckling load, followed by the cylindrical shell ($a = \infty$), whereas the concave

toroidal shell segment ($a = -4R$) gives the lowest values over the entire temperature range. For example, at $T = 300$ K, the static buckling loads are 2.544 MPa for the convex shell, 1.480 MPa for the cylindrical shell, and only 0.852 MPa for the concave shell. A similar order is maintained for the dynamic buckling loads at both loading rates. Another important observation is that the dynamic buckling loads are consistently higher than the corresponding static values for all shell types. This difference becomes more noticeable as the loading rate increases from 10^6 to 2×10^6 Pa/s, confirming the significant role of inertial effects under rapidly increasing pressure. For instance, for the convex shell at $T = 400$ K, the buckling load increases from 2.268 MPa in the static case to 2.308 MPa and 2.328 MPa for the two dynamic loading rates, respectively. The same trend is also seen for the cylindrical shell and the concave shell.

Table 2 also indicates that temperature has a

detrimental effect on buckling capacity. As the temperature increases from 300 K to 500 K, both static and dynamic buckling loads decrease for all geometrical configurations. For the convex shell, the static buckling load decreases from 2.544 MPa at 300 K to 2.166 MPa at 500 K. Likewise, the corresponding values for the cylindrical shell decrease from 1.480 MPa to 1.266 MPa, while those for the concave shell reduce from 0.852 MPa to 0.727 MPa. This reduction can be attributed to the degradation of the effective stiffness of FG-GRC materials at elevated temperatures. In addition, the buckling mode remains almost unchanged for each shell type as the loading rate varies, whereas slight changes may occur with increasing temperature. The convex shell is mainly associated with the mode (1;6;11;74.78) at 300 K and 400 K, which changes slightly to (1;6;12;73.35) at 500 K. A similar trend is observed for the cylindrical and concave shells.

Table 2. Comparison of static and dynamic buckling loads for FG-GRC cylindrical shells and toroidal shell segments under different geometric configurations (MPa, UD-PC-UD/UD, $h = 2$ mm, $h_{co} = 80h$, $e_1 = 0.3$, $R = 80h$, $L = 1.5R$, $b_{sp} = h$, $h_{sp} = 1.5h$, $d_{sp} = 24$ mm)

	T(K)	Static	Dynamic	
			$q_0 = 10^6$ (Pa/s)	$q_0 = 2 \times 10^6$ (Pa/s)
Convex shell ($a = 4R$)	300	2.544 (1;6;11;74.78) ^a	2.563 (1;6;11;74.78)	2.572 (1;6;11;74.78)
	400	2.268 (1;6;11;74.78)	2.308 (1;6;11;74.78)	2.328 (1;6;11;74.78)
	500	2.166 (1;6;12;73.35)	2.202 (1;6;12;73.35)	2.222 (1;6;12;73.35)
Cylindrical shell ($a = \infty$)	300	1.480 (1;5;14;70.47)	1.500 (1;5;14;70.47)	1.514 (1;5;14;70.47)
	400	1.323 (1;5;14;70.47)	1.360 (1;5;14;70.47)	1.391 (1;5;14;70.47)
	500	1.266 (1;5;15;69.02)	1.346 (1;5;15;69.02)	1.433 (1;5;15;69.02)
Concave shell ($a = -4R$)	300	0.852 (1;4;18;64.55)	0.877 (1;4;18;64.55)	0.892 (1;4;18;64.55)
	400	0.760 (1;4;18;64.55)	0.806 (1;4;18;64.55)	0.839 (1;4;18;64.55)
	500	0.727 (1;4;18;64.55)	0.765 (1;4;18;64.55)	0.787 (1;4;18;64.55)

^a The buckling modes ($m; n; n_{sp}; \chi$)

Table 3 shows that both the stiffener configuration and graphene distribution have significant effects on the static and dynamic buckling loads of convex hollow-core FG-GRC

toroidal shell segments. For all material models, the unstiffened shells exhibit the lowest buckling loads, the orthogonally stiffened shells provide a substantial improvement, and the helically stiffened

shells give the highest values. For example, in the UD-PC-UD/UD configuration, the static buckling load increases from 0.724 MPa for the unstiffened shell to 1.775 MPa for the orthogonally stiffened shell and further to 2.268 MPa for the helically stiffened shell. The same trend is maintained for the dynamic cases at both loading rates. The results also indicate that the graphene distribution strongly influences the buckling resistance. Among the three investigated material configurations, UD-PC-UD/UD gives the highest buckling loads, followed by X-PC-X/X, while O-PC-O/O yields the

lowest values for all stiffener cases. In addition, the dynamic buckling loads are consistently higher than the corresponding static values, and they increase as the loading rate rises from 10^6 to 2×10^6 Pa/s, confirming the beneficial role of inertial effects under rapidly increasing pressure. Finally, the buckling mode is affected by the stiffener type. The unstiffened shells are predominantly associated with the mode (1;8), whereas the orthogonally and helically stiffened shells generally exhibit different buckling modes, mainly with lower circumferential wave numbers.

Table 3. Effects of stiffener configurations and graphene distribution patterns on the dynamic buckling loads of FG-GRC toroidal shell segments (MPa, $a = 4R$, $h = 2\text{mm}$, $h_{co} = 80h$, $e_1 = 0.3$, $T = 400\text{K}$, $R = 80h$, $L = 1.5R$, $b_{ri} = b_{st} = b_{sp} = h$, $h_{ri} = h_{st} = h_{sp} = 1.5h$, $d_{ri} = d_{st} = d_{sp} = 24\text{mm}$, $n_{ri} = 10$, $n_{st} = 42$)

	Stiffener types	Static	Dynamic	
			$c = 10^6(\text{Pa/s})$	$c = 2 \times 10^6(\text{Pa/s})$
UD-PC-UD/UD	Unstiffener	0.724 (1;8) ^a	0.739 (1;8)	0.798 (1;8)
	Orthogonal stiffener	1.775 (1;6)	1.808 (1;6)	1.825 (1;6)
	Helical stiffener	2.268 (1;6;11;74.78) ^b	2.308 (1;6;11;74.78)	2.328 (1;6;11;74.78)
X-PC-X/X	Unstiffener	0.677 (1;8)	0.692 (1;8)	0.701 (1;8)
	Orthogonal stiffener	1.665 (1;6)	1.695 (1;6)	1.713 (1;6)
	Helical stiffener	2.130 (1;6;11;74.78)	2.166 (1;6;11;74.78)	2.188 (1;6;11;74.78)
O-PC-O/O	Unstiffener	0.646 (1;8)	0.661 (1;8)	0.669 (1;8)
	Orthogonal stiffener	1.592 (4;7)	1.621 (4;7)	1.638 (4;7)
	Helical stiffener	2.011 (1;6;12;73.35)	2.046 (1;6;12;73.35)	2.070 (1;6;12;73.35)

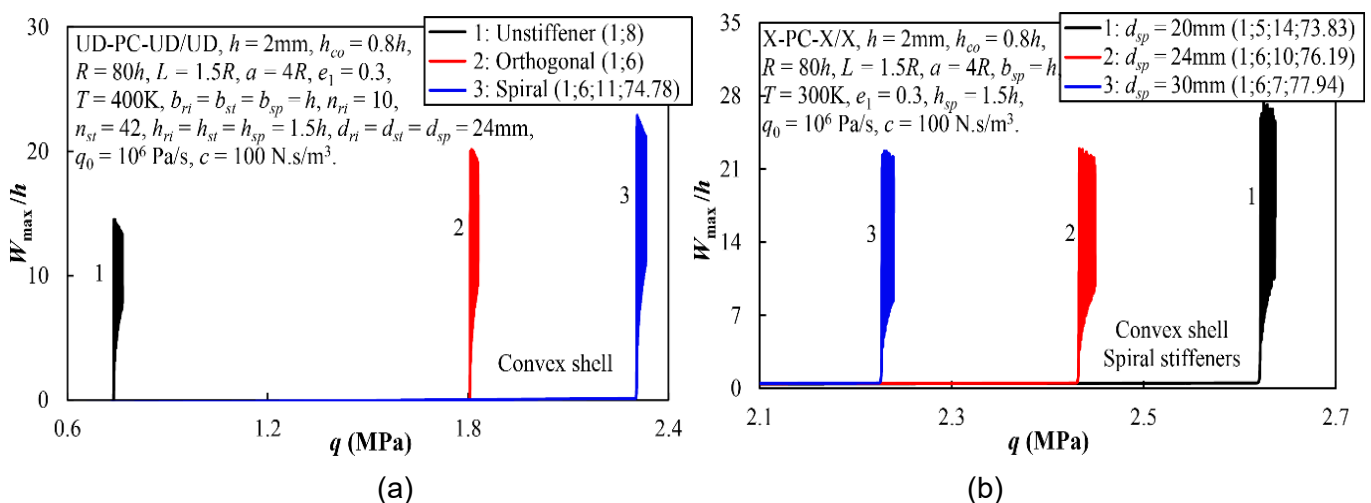


Fig. 2. Effects of stiffeners and stiffener spacing on the dynamic buckling behavior of toroidal shell segments. (a) Effects of stiffeners; (b) Effects of stiffener spacings

Fig. 2 illustrates the effects of stiffener type and stiffener spacing on the dynamic buckling behavior of convex toroidal shell segments. As shown in Fig. 2(a), the dynamic buckling resistance is markedly improved by the presence of stiffeners. The unstiffened shell exhibits the lowest buckling load, whereas the orthogonally stiffened shell shows a significant increase, and the helically stiffened shell provides the highest resistance. This confirms that helical stiffeners are more effective in enhancing shell stability under rapidly increasing external pressure. Fig. 2(b) further shows that the stiffener spacing has a pronounced influence on the dynamic buckling load. As the spacing decreases from $d_{sp} = 24$ mm to 22 mm and then to 20 mm, the buckling load increases noticeably. This trend indicates that a denser stiffening layout

leads to greater overall stiffness and improved resistance to dynamic instability. Therefore, both the stiffener configuration and spacing play essential roles in the dynamic buckling performance of FG-GRC toroidal shell segments, with closely spaced helical stiffeners giving the most favorable response.

Fig. 3 illustrates the influence of shell geometry on the dynamic buckling behavior of toroidal shell segments. As shown in Fig. 3(a), shell type has a pronounced effect on buckling resistance. The convex shell exhibits the highest dynamic buckling load, followed by the cylindrical shell, whereas the concave shell gives the lowest value. This confirms that positive meridional curvature is beneficial for enhancing structural stability under rapidly increasing external pressure.

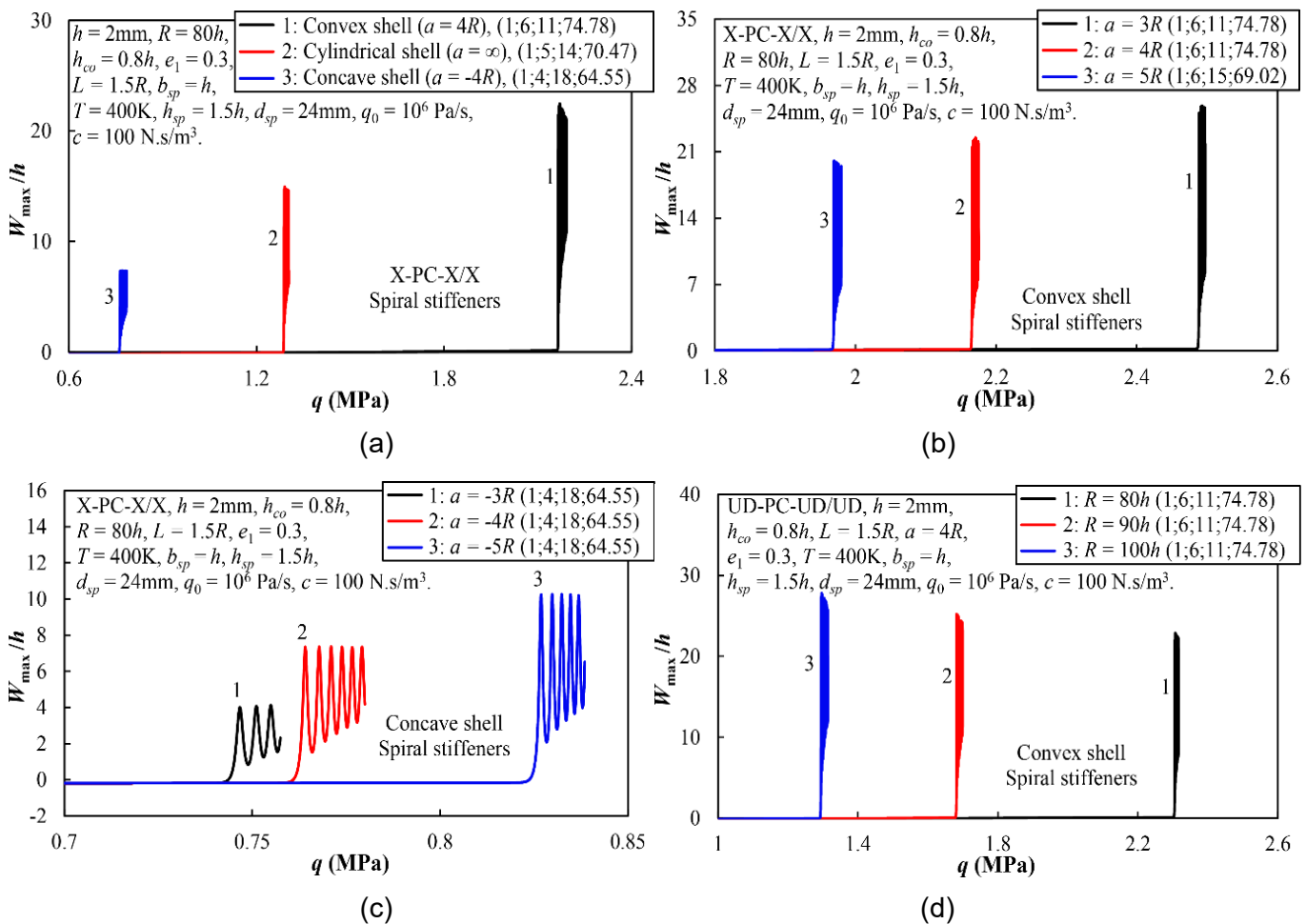


Fig. 3. Effects of geometrical properties of shells on the dynamic buckling behavior of toroidal shell segments. (a) Effects of shell types; (b) Effects of meridian radius (convex shells); (c) Effects of meridian radius (concave shells); (d) Effects of circumferential radius

Figs. 3(b) and 3(c) further indicate that the meridional radius significantly affects the dynamic buckling response. For convex shells, increasing the meridional radius from $a = 3R$ to $5R$ leads to a noticeable decrease in the dynamic buckling load, indicating that stronger meridional curvature is beneficial to dynamic stability. In contrast, for concave shells, the influence of the meridional radius is also evident, although the buckling loads

remain much lower than those of convex shells. This result suggests that concave curvature is inherently less favorable from the viewpoint of dynamic buckling resistance. Fig. 3(d) shows that the circumferential radius also plays an important role. As R increases, the dynamic buckling load decreases, indicating that shells with smaller circumferential radius possess greater stiffness and higher resistance to dynamic instability.

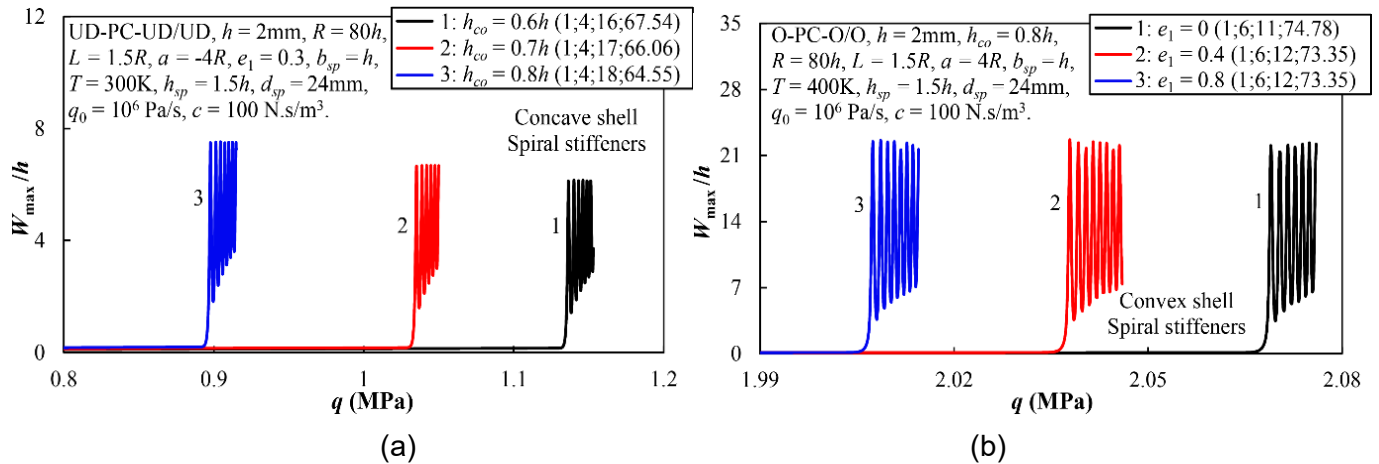


Fig. 4. Effects of core thickness and porosity coefficient on the dynamic buckling behavior of toroidal shell segments. (a) Effects of core thicknesses; (b) Effects of porosity coefficients

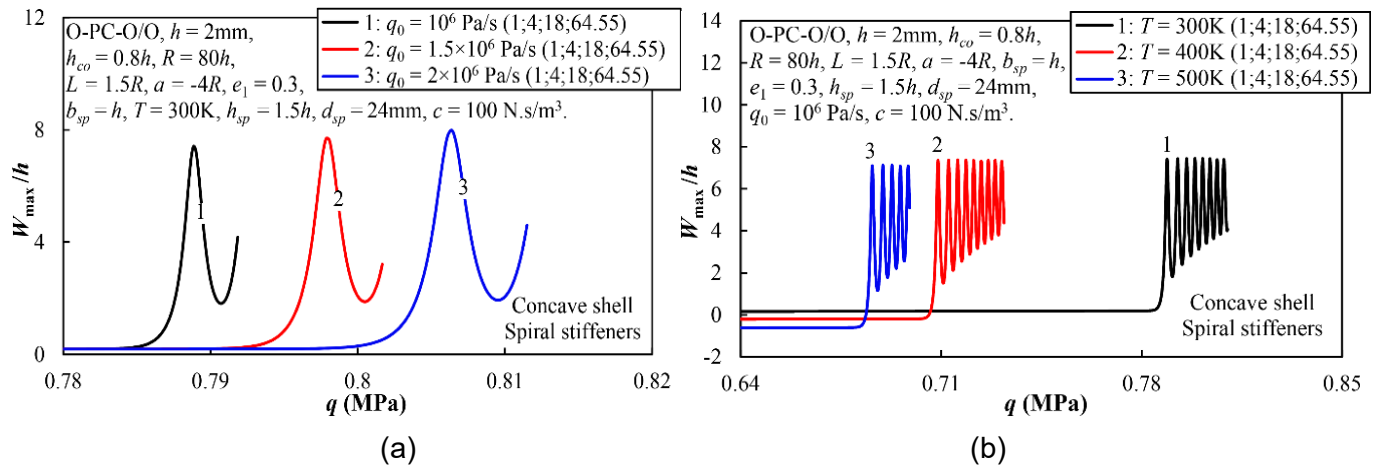


Fig. 5. Effects of loading velocity and temperature on the dynamic buckling behavior of toroidal shell segments. (a) Effects of loading velocities; (b) Effects of temperatures

Fig. 4 presents the effects of core thickness and porosity coefficient on the dynamic buckling behavior of toroidal shell segments. In Fig. 4(a), increasing the core thickness ratio leads to a clear reduction in the dynamic buckling load. The curves shift toward lower pressure levels as the core becomes thicker, indicating that although the shell becomes lighter, the reduction in effective stiffness

dominates and weakens the resistance to dynamic buckling. Fig. 4(b) shows that the porosity coefficient also has a detrimental effect on structural stability. As the porosity coefficient increases, the dynamic buckling load decreases noticeably. This trend can be attributed to the loss of effective material stiffness caused by the higher void content in the core layer. Therefore, both a

thicker porous core and a larger porosity coefficient reduce the dynamic buckling resistance of the shell.

Fig. 5 illustrates the effects of loading velocity and temperature on the dynamic buckling behavior of toroidal shell segments. As shown in Fig. 5(a), increasing the loading velocity significantly enhances the dynamic buckling resistance. The critical pressure shifts to higher values as the loading rate increases, indicating the important role of inertial effects under rapidly applied loads. Fig. 5(b) shows that temperature has an opposite influence. As the temperature increases from 300 K to 500 K, the dynamic buckling load decreases noticeably. This reduction is attributed to the degradation of the effective stiffness of FG-GRC materials at elevated temperatures.

Fig. 6 shows the effects of graphene

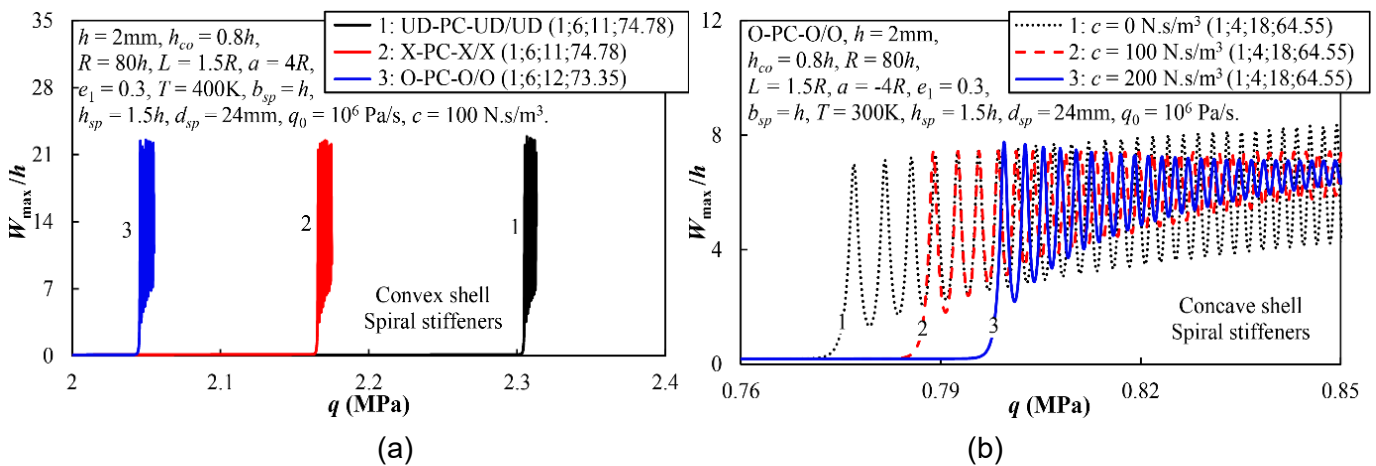


Fig. 6. Effects of graphene distribution law and damping coefficient on the dynamic buckling behavior of toroidal shell segments. (a) Effects of graphene distribution laws; (b) Effects of damping coefficients

6. Conclusion

An analytical formulation has been developed to investigate the nonlinear large-deflection dynamic buckling of hollow-core FG-GRC circular toroidal shell segments reinforced by orthogonal and helical stiffeners under rapidly increasing external pressure. The model was established on the basis of Donnell's shell theory with von Kármán geometric nonlinearity, in which the effects of porous core configuration, graphene distribution, thermal environment, and stiffener arrangement were taken into account. By

distribution law and damping coefficient on the dynamic buckling behavior of toroidal shell segments. As seen in Fig. 6(a), the graphene distribution pattern has a clear influence on the dynamic buckling load. Among the considered cases, the UD-PC-UD/UD configuration provides the highest buckling resistance, followed by X-PC-X/X, while O-PC-O/O gives the lowest value. This indicates that the distribution of graphene through the thickness plays an important role in tailoring the stiffness and stability of the shell. Fig. 6(b) illustrates the effect of the damping coefficient. As the damping coefficient increases from $c = 0$ to 100 and 200 N.s/m^3 , the dynamic buckling load shifts to higher values. This trend shows that damping suppresses vibration amplitudes and delays the onset of instability under rapidly increasing external pressure.

combining the smeared stiffener technique, the Airy stress function, the Galerkin procedure, and the Runge-Kutta method, the dynamic buckling response of the shell segments was determined, while the dynamic buckling load was evaluated using the Budiansky-Roth criterion.

The numerical results showed that shell geometry has a pronounced effect on dynamic stability. Convex toroidal shell segments possess the highest buckling resistance, whereas concave shells exhibit the lowest one. In addition, the dynamic buckling loads are always higher than the

corresponding static values and further increase with loading rate, confirming the important role of inertial effects under transient pressure. By contrast, elevated temperature significantly reduces both static and dynamic buckling loads due to the degradation of the effective stiffness of FG-GRC materials.

The results also confirmed the beneficial influence of stiffening and material design. Helical stiffeners provide the greatest improvement in buckling resistance, followed by orthogonal stiffeners, while unstiffened shells remain the weakest configuration. A reduction in stiffener spacing further enhances dynamic stability. Among the considered graphene distributions, the UD-PC-UD/UD model yields the highest buckling loads, followed by X-PC-X/X and O-PC-O/O. Moreover, increasing the porous core thickness or porosity coefficient decreases the dynamic buckling resistance, whereas larger damping coefficients improve structural stability by delaying the onset of instability.

Declaration of Generative AI

During the preparation of this work, the authors did not use Generative AI and AI-assisted technology tools in order to generate the content of the manuscript.

References

- [1]. K. Gao, W. Gao, D. Wu, C. Song. (2018). Nonlinear dynamic buckling of the imperfect orthotropic E-FGM circular cylindrical shells subjected to the longitudinal constant velocity. *International Journal of Mechanical Sciences*, 138-139, 199-209. <https://doi.org/10.1016/j.ijmecsci.2018.02.013>
- [2]. X. Liang, X. Zha, Y. Yu, Z. Cao, X. Jiang, J. Leng. (2019). Semi-analytical vibration analysis of FGM cylindrical shells surrounded by elastic foundations in a thermal environment. *Composite Structures*, 223, 110997. <https://doi.org/10.1016/j.compstruct.2019.110997>
- [3]. M. Ramezani, M. Rezaiee-Pajand, F. Tornabene. (2022). Nonlinear dynamic analysis of FG/SMA/FG sandwich cylindrical shells using HSDT and semi ANS functions. *Thin-Walled Structures*, 171, 108702. <https://doi.org/10.1016/j.tws.2021.108702>
- [4]. H.S. Shen. (2005). Postbuckling of axially loaded FGM hybrid cylindrical shells in thermal environments. *Composites Science and Technology*, 65(11-12), 1675-1690. <https://doi.org/10.1016/j.compscitech.2005.02.008>
- [5]. H.S. Shen, N. Noda. (2007). Postbuckling of pressure-loaded FGM hybrid cylindrical shells in thermal environments. *Composite Structures*, 77(4), 546-560. <https://doi.org/10.1016/j.compstruct.2005.08.006>
- [6]. B. Ni, L. Xi. (2025). Stochastic Vibration Responses of Functionally Graded Three-Stage Conical–Cylindrical Combined Shell Structure Subjected to Various Stationary Stochastic Excitations. *International Journal of Structural Stability and Dynamics*, 25(20), 2550211. <https://doi.org/10.1142/S0219455425502116>
- [7]. Z. Cao, Y. Hu. (2026). Magneto-Thermoelastic Principal Parameter Resonance of a Functionally Graded Cylindrical Shell with Axial Tension. *International Journal of Structural Stability and Dynamics*, 26(01), 2550256. <https://doi.org/10.1142/S0219455425502566>
- [8]. V.H. Nam, N.T. Phuong, K.V. Minh, P.T. Hieu. (2018). Nonlinear thermo-mechanical buckling and post-buckling of multilayer FGM cylindrical shell reinforced by spiral stiffeners surrounded by elastic foundation subjected to torsional loads. *European Journal of Mechanics - A/Solids*, 72, 393-406. <https://doi.org/10.1016/j.euromechsol.2018.06.005>
- [9]. H.S. Shen. (2011). Postbuckling of nanotube-reinforced composite cylindrical shells in thermal environments, Part I: Axially-loaded

- shells. *Composite Structures*, 93(8), 2096-2108.
<https://doi.org/10.1016/j.compstruct.2011.02.011>
- [10]. H.S. Shen. (2011). Postbuckling of nanotube-reinforced composite cylindrical shells in thermal environments, Part II: Pressure-loaded shells. *Composite Structures*, 93(10), 2496-2503.
<https://doi.org/10.1016/j.compstruct.2011.04.005>
- [11]. H. Asadi, Q. Wang. (2017). Dynamic stability analysis of a pressurized FG-CNTRC cylindrical shell interacting with supersonic airflow. *Composites Part B: Engineering*, 118, 15-25.
<https://doi.org/10.1016/j.compositesb.2017.03.001>
- [12]. S. Sun, C. Guo, W. Feng, D. Cao. (2022). Nonlinear vibration analysis of CNT-reinforced functionally graded composite cylindrical shells resting on elastic foundations. *International Journal of Non-Linear Mechanics*, 143, 104037.
<https://doi.org/10.1016/j.ijnonlinmec.2022.104037>
- [13]. Y. Kiani, R. Dimitri, F. Tornabene. (2018). Free vibration of FG-CNT reinforced composite skew cylindrical shells using the Chebyshev-Ritz formulation. *Composites Part B: Engineering*, 147, 169-177.
<https://doi.org/10.1016/j.compositesb.2018.04.028>
- [14]. R. Ansari, E. Hasrati, J. Torabi. (2019). Vibration analysis of pressurized sandwich FG-CNTRC cylindrical shells based on the higher-order shear deformation theory. *Materials Research Express*, 6(4), 045049.
<https://doi.org/10.1088/2053-1591/aafcb7>
- [15]. D.T. Dong, V.H. Nam, N.T. Phuong, L.N. Ly, V.M. Duc, N.V. Tien, T.Q. Minh, V.T. Hung, P.H. Quan. (2022). An analytical approach of nonlinear buckling behavior of longitudinally compressed carbon nanotube-reinforced (CNTR) cylindrical shells with CNTR stiffeners in thermal environment. *ZAMM - Journal of Applied Mathematics and Mechanics*, 102(4), e202100228.
<https://doi.org/10.1002/zamm.202100228>
- [16]. H.S. Shen, Y. Xiang. (2018). Postbuckling of functionally graded graphene-reinforced composite laminated cylindrical shells subjected to external pressure in thermal environments. *Thin-Walled Structures*, 124, 151-160.
<https://doi.org/10.1016/j.tws.2017.12.005>
- [17]. H.S. Shen, Y. Xiang. (2018). Postbuckling behavior of functionally graded graphene-reinforced composite laminated cylindrical shells under axial compression in thermal environments. *Computer Methods in Applied Mechanics and Engineering*, 330, 64-82.
<https://doi.org/10.1016/j.cma.2017.10.022>
- [18]. H.S. Shen, Y. Xiang. (2019). Torsional postbuckling behavior of FG-GRC laminated cylindrical shells in thermal environments. *Thin-Walled Structures*, 135, 560-574.
<https://doi.org/10.1016/j.tws.2018.11.025>
- [19]. Z.M. Li, T. Liu, D. Yang, P. Qiao. (2024). Postbuckling behaviour of anisotropic shear deformable laminated cylindrical shells with general imperfection distribution subjected to torsion. *Thin-Walled Structures*, 204, 112292.
<https://doi.org/10.1016/j.tws.2024.112292>
- [20]. N.T. Phuong, N.T. Trung, C.V. Doan, N.D. Thang, V.M. Duc, V.H. Nam. (2020). Nonlinear thermomechanical buckling of FG-GRC laminated cylindrical shells stiffened by FG-GRC stiffeners subjected to external pressure. *Acta Mechanica*, 231, 5125-5144.
<https://doi.org/10.1007/s00707-020-02813-5>
- [21]. H.S. Shen, Y. Xiang, Y. Fan. (2017). Nonlinear vibration of functionally graded graphene-reinforced composite laminated cylindrical shells in thermal environments. *Composite Structures*, 182, 447-456.
<https://doi.org/10.1016/j.compstruct.2017.09.0>

10

- [22]. Z.X. Lei, L.H. Tong. (2019). Analytical solution of low-velocity impact of graphene-reinforced composite functionally graded cylindrical shells. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 41, 486. <https://doi.org/10.1007/s40430-019-1983-5>
- [23]. D.T. Dong, D.T.K. My, V.M. Duc, L.N. Ly, V.T. Hung, V.H. Nam. (2024). Nonlinear torsional buckling of corrugated core sandwich toroidal shell segments with graphene-reinforced coatings in temperature change using the Ritz energy method. *Applied Mathematical Modelling*, 126, 739-752. <https://doi.org/10.1016/j.apm.2023.11.027>
- [24]. N.T. Phuong, C.V. Doan, V.M. Duc, N.T. Giang, V.H. Nam. (2023). Analytical solution for nonlinear buckling of convex and concave auxetic-core toroidal shell segments with graphene-reinforced face sheets subjected to radial loads. *Archive of Applied Mechanics*, 93, 621-634. <https://doi.org/10.1007/s00419-022-02288-x>
- [25]. Y. Kiani. (2019). Buckling of functionally graded graphene reinforced conical shells under external pressure in thermal environment. *Composites Part B: Engineering*, 156, 128-137. <https://doi.org/10.1016/j.compositesb.2018.08.052>
- [26]. F. Abbaspour, H. Arvin, Y. Kiani. (2022). Mechanical buckling analysis of functionally graded composite laminated plates reinforced with temperature dependent graphene sheets resting on elastic foundation. *ZAMM - Journal of Applied Mathematics and Mechanics*, 102(1), e202100097. <https://doi.org/10.1002/zamm.202100097>
- [27]. T.Q. Minh, V.H. Nam, V.M. Duc, V.T. Hung, L.N. Ly, N.T. Phuong. (2024). Nonlinear vibration and dynamic buckling responses of stiffened functionally graded graphene-reinforced cylindrical, parabolic, and sinusoid panels using the higher-order shear deformation theory. *ZAMM - Journal of Applied Mathematics and Mechanics*, 104(3), e202300580. <https://doi.org/10.1002/zamm.202300580>
- [28]. L.N. Quang, V.M. Duc, T.Q. Minh. (2024). Nonlinear buckling and postbuckling behaviors of porous FG-GPLRC cylindrical shells with stiffeners subjected to external pressure. *Journal of Science and Transport Technology* 4(2), 24-36. <https://doi.org/10.58845/jstt.utt.2024.en.4.2.24-36>
- [29]. K.Q. Thai, D.H. Bich, N.T. Phuong, C.V. Doan, T.Q. Minh. (2025). Nonlinear buckling behavior of FG-GPLRC cylindrical shell stiffened by spiral FG-GPLRC stiffeners subjected to axial compressive load in thermal environment. *Vietnam Journal of Mechanics*, 48(2), 186-201. <https://doi.org/10.15625/0866-7136/23703>
- [30]. V.H. Nam, K.L. Huong, L.N. Ly, C.V. Doan, P.T. Hieu. (2026). Nonlinear buckling analysis of porous-core FG-GPLRC toroidal shell segments with generalized meridional curvature reinforced by orthogonal stiffeners. *Vietnam Journal of Mechanics*, 48(3), 268-281. <https://doi.org/10.15625/0866-7136/23994>
- [31]. D.T.K. My, K.L. Huong, N.V. Tien. (2026). Analytical study on nonlinear buckling of spiral-stiffened FG-CNTRC toroidal shells segments with piezoelectric layers and generalized curvature under external pressure. *Journal of Science and Transport Technology*, 6(2), 75-97. <https://doi.org/10.58845/jstt.utt.2026.en.6.2.75-97>