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Nonlinear free vibration of FG-GPLRPC conical shells reinforced by circular FG-GPLRPC stiffeners in thermal environment

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Abstract: This paper presents a comprehensive analytical study on the nonlinear free vibration behavior of functionally graded graphene platelet-reinforced porous composite (FG-GPLRPC) conical shells integrated with circular FG-GPLRPC ring stiffeners under thermal conditions. The theoretical formulation is established within the framework of Donnell shell theory, incorporating von Kármán geometric nonlinearity to account for large deflections. The stiffening effect of circular reinforcements is modeled using an enhanced Lekhnitskii-based smeared stiffener approach. By employing the Ritz method in conjunction with the harmonic balance technique, the nonlinear governing equations are derived and subsequently used to determine the natural frequencies and amplitude–frequency relationships. A detailed parametric investigation is conducted to evaluate the effects of stiffener characteristics, graphene platelet content, porosity distribution, material gradation patterns, and geometric parameters on the nonlinear vibration response. The results provide valuable insights into the dynamic behavior and design optimization of advanced porous graphene-reinforced conical shell structures operating in thermal environments.

Keywords: Nonlinear free vibration, Functionally graded graphene platelet-reinforced porous composite, Donnell shell theory, Circular stiffener, Ritz energy method, Conical shell.

1. Introduction

Conical shells are widely used in various engineering applications due to their high structural efficiency, and their dynamic behavior has been extensively studied [1, 2]. In recent years, functionally graded materials (FGMs) have attracted significant attention for such structures, particularly in high-temperature environments. Owing to the continuous variation in material

composition, FGMs exhibit smoothly graded properties, which effectively reduce interfacial debonding and stress concentration compared to conventional composites.

Linearized stability equations for the mechanical buckling of FGM conical panels were formulated by Bich et al. [3] within the framework of classical shell theory, and explicit critical loads were obtained through the Galerkin approach.

Regarding vibration problems, the free vibration response of FGM conical panels subjected to thermal environments was investigated by Jooybar et al. [4] using the differential quadrature method, where the results revealed a pronounced dependence on geometrical parameters as well as material gradation. Zhao et al. [5] further explored the vibration characteristics of such structures through an element-free KP-Ritz approach, providing detailed insights into the effects of physical and geometrical parameters. Additional contributions have been reported for conical and related shell structures. Xie et al. [6] presented analytical solutions for the free vibration of FGM annular plates and conical shells, while Akbari et al. [7] investigated the vibration response of Lévy-type FGM conical panels using the generalized differential quadrature method. The nonlinear vibration behavior of FGM conical shells embedded in elastic media was studied by Sofiyev et al. [8], highlighting the role of foundation effects. Moreover, Xie et al. [9] analyzed both free and forced vibrations of underwater ring-stiffened conical shells under various boundary conditions using an analytical approach. Further studies have extended the analysis of functionally graded conical shells to more complex loading and multiphysical conditions. Malekzadeh et al. [10] investigated the vibration behavior of FGM conical shells subjected to asymmetric moving loads, revealing the significant influence of dynamic excitation patterns. Hao et al. [11] examined the aero-thermo-elastic flutter characteristics of ceramic-metal graded truncated conical shells under combined aerodynamic pressure and thermal loading. To improve the accuracy of structural modeling, first-order shear deformation theory (FSDT) has been widely employed. Malekzadeh et al. [12-14] utilized FSDT to study the free vibration of FGM shells, demonstrating its effectiveness in capturing transverse shear effects. Similarly, Sofiyev and Osmancelebioglu [15] analyzed the vibration of sandwich truncated

conical shells with FGM layers within the FSDT framework combined with Donnell-type kinematics. Asanjarani et al. [16] further investigated the free vibration behavior of FG truncated conical shells using Hamilton's principle and the differential quadrature method. In addition, the vibration and stability responses of conical shells under various loading and boundary conditions have been extensively examined. Sofiyev [17] studied the dynamic and stability behavior of freely supported FGM truncated conical shells subjected to uniform lateral and hydrostatic pressures. A comprehensive overview of the buckling and vibration characteristics of functionally graded conical shell structures, including layered and sandwich configurations, was provided by Sofiyev [18], offering valuable insights for subsequent developments in this field.

Functionally graded graphene platelet-reinforced composite (FG-GPLRC) has recently attracted significant attention due to the outstanding mechanical and thermal properties imparted by graphene, combined with the inherent advantages of polymer matrices, such as high deformability, flexibility, and chemical stability. As a result, numerous studies have explored the mechanical and thermal responses of FG-GPLRC shell structures under various loading conditions [19-21]. More recently, the incorporation of porosity into FG-GPLRC has led to the development of functionally graded graphene platelet-reinforced porous composite (FG-GPLRPC). By introducing controlled pore distributions through different gradation patterns, these materials achieve a balance between reduced weight and enhanced mechanical performance. The combination of graphene reinforcement and porous architecture provides high specific stiffness, improved thermal conductivity, and considerable potential for applications in lightweight structures, energy absorption systems, and thermal management devices [22].

Recent studies have increasingly focused on

the mechanical and dynamic behavior of FG-GPLRC conical shells. Thai et al. [23] investigated the nonlinear thermal buckling and postbuckling responses of FG-GPLRC conical shells using Donnell shell theory in conjunction with the Ritz energy method. Extending this line of research, Safarpour et al. [24] employed three-dimensional elasticity theory together with the generalized differential quadrature method to analyze both static and free vibration behaviors of GPLRC structures, including truncated conical shells, cylindrical shells, and annular plates. The vibration characteristics of GPLRPC conical shells have also been examined using refined theoretical models. Khayat et al. [25] adopted higher-order shear deformation theory combined with Sanders' shell theory and the Fourier differential quadrature method to study the free vibration response of GPLRPC truncated conical shells. Furthermore, Cho [26] developed a meshfree-based nonlinear approach to investigate the nonlinear free vibration of FG-GPLRC conical panels on elastic foundations. The formulation was based on first-order shell theory with von Kármán geometric nonlinearity, and the resulting nonlinear governing equations were solved using an efficient iterative scheme. More recent investigations have focused on the nonlinear dynamic behavior of FG-GPLRC conical shell structures under various complex conditions. Saboori et al. [27] analyzed the instability and nonlinear responses of FG-GPLRPC conical shells considering internal resonance using perturbation-based techniques. Hanet et al. [28] investigated the forced vibration behavior of sandwich conical shells with electrorheological cores and FG-GPLRC face sheets, highlighting the coupled effects of smart materials and graphene-based reinforcement on the dynamic response. Zhang et al. [29] investigated the nonlinear dynamic behavior of axially moving conical shells made of FG-GPLRPC under thermal environments, with particular emphasis on combined resonance and complex nonlinear

phenomena. Bahrnifard et al. [30] investigated the dynamic response of ring-stiffened sandwich FG-GPLRC conical shells with porous cores under moving loads. Zhang et al. [31] examined the effects of partial elastic foundations on the thermally induced nonlinear vibration of FG-GPLRC conical shells. Moreover, Yang et al. [32] analyzed nonlinear dynamic responses of sandwich truncated conical shells with variable thickness, including internal resonance phenomena. Beyond conical shell structures, Tu et al. [33] and Phuong et al. [34] extended the analysis to spherical caps, investigating nonlinear buckling and postbuckling behaviors using energy-based methods.

Despite the extensive body of research on the vibration and stability of functionally graded and graphene-reinforced conical shell structures, several important issues remain insufficiently addressed. Most existing studies have primarily focused on either conventional FGMs or graphene platelet-reinforced composites without considering the combined effects of porosity and functional gradation. Although some works have incorporated porous graphene-reinforced materials or advanced multiphysical effects, the nonlinear vibration behavior of such structures has not been systematically investigated, particularly for conical shell configurations. Furthermore, the influence of structural reinforcements, such as ring stiffeners, has received limited attention in the context of graphene-reinforced porous composites, despite their practical importance in enhancing structural stiffness and stability.

To address these gaps, the present study develops an analytical framework to investigate the nonlinear free vibration of FG-GPLRPC conical shells reinforced with circular FG-GPLRPC stiffeners under thermal conditions. The combined effects of graphene reinforcement and porosity provide a unique balance between stiffness enhancement and mass reduction, which plays an important role in nonlinear vibration behavior. In

particular, these material characteristics can significantly influence the frequency–amplitude relationship through their impact on structural stiffness and inertia distribution. The formulation is based on Donnell shell theory combined with von Kármán geometric nonlinearity, while the stiffening effect is incorporated using an improved smeared stiffener technique. The governing equations are

derived via the Ritz method and solved using the harmonic balance approach to obtain the frequency–amplitude relationships. The influences of key parameters, including porosity, graphene content, stiffener configuration, and geometric characteristics, are systematically examined to provide new insights into the design of advanced lightweight conical shell structures.

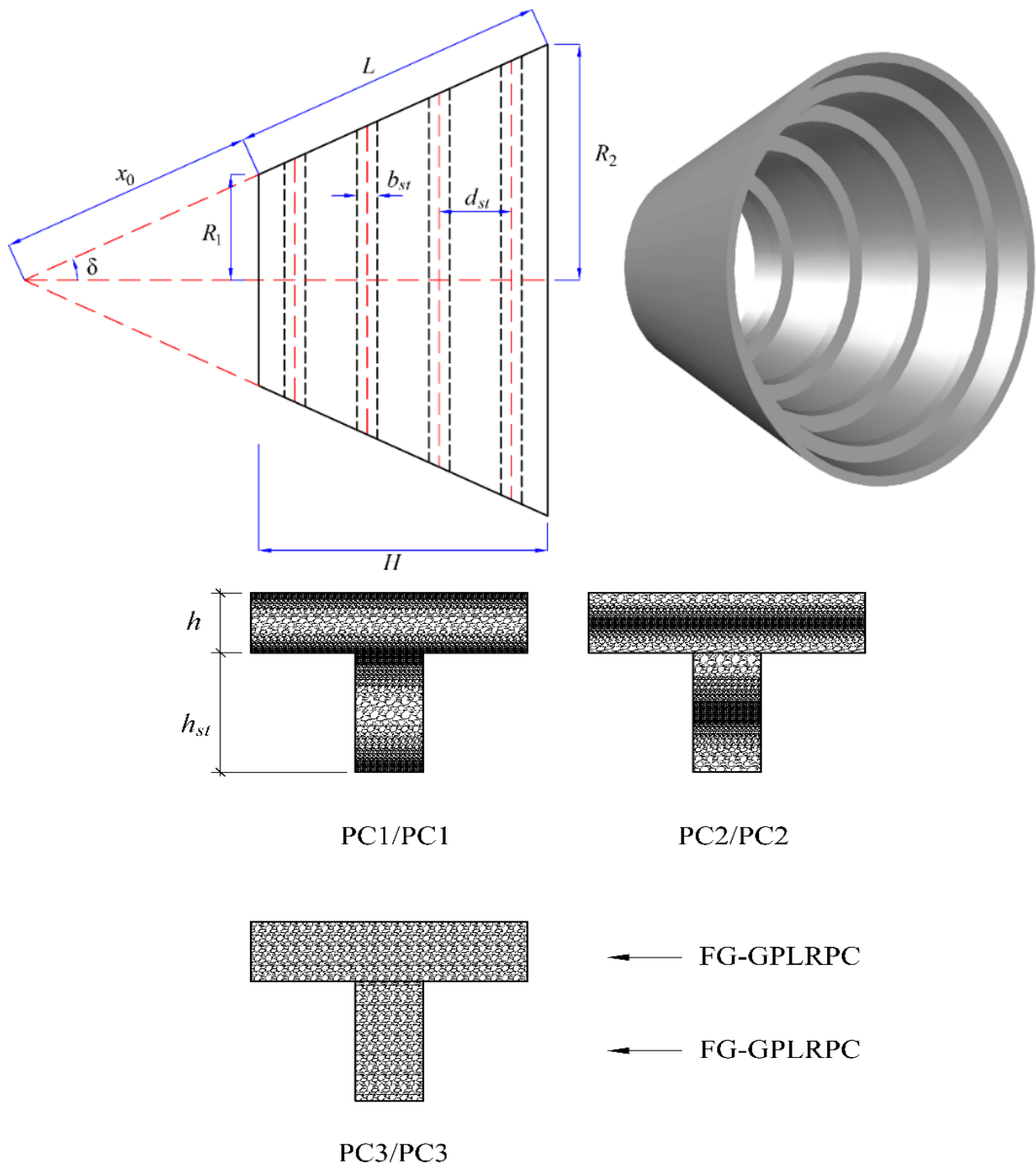


Fig. 1. Geometry and coordinate system of the FG-GPLRPC conical shell with ring stiffeners and porosity distributions.

2. Material and Geometry Modeling of FG-GPLRPC Conical Shells

Consider a functionally graded graphene platelet-reinforced porous composite (FG-GPLRPC) conical shell, as illustrated in Fig. 1. The shell has a thickness h , slant length L , axial length H , and semi-vertex angle δ , with R_1 and R_2 denoting the radii of the small and large bases, respectively. The structure is defined in a curvilinear coordinate system (x, θ, z) , where the origin is located at the mid-surface. The parameter x_0 represents the distance from the apex to the small base. The displacement components of a point on the mid-surface are denoted by u, v , and w along the x, θ and z directions, respectively.

As illustrated in Fig. 1, the porosity distributions PC1 and PC2 are symmetric with respect to the mid-surface along the thickness direction. In the PC1 configuration, the maximum pore size is located at the mid-surface, whereas in PC2, the minimum pore size occurs at this location. In contrast, the porosity in PC3 is uniformly distributed throughout the thickness. In this study, the stiffeners and FG-GPLRPC layers are considered with identical porosity patterns, leading to three configurations denoted as PC1/PC1, PC2/PC2, and PC3/PC3.

The effective material properties of the shell and stiffeners, including Poisson's ratio ν , thermal expansion coefficient α , mass density ρ , and Young's modulus E , are evaluated within the thickness domain $(-h/2 \leq z \leq h/2)$ as follows:

$$\bar{\nu}_{pc}^{sh} = \bar{\nu}_{sh}, \bar{\alpha}_{pc}^{sh} = \bar{\alpha}_{sh}, \quad (1)$$

$$\bar{\rho}_{pc}^{sh} = \begin{cases} \bar{\rho}_{sh} \left[1 - \bar{e}_{1sh} \cos\left(\frac{\pi z}{h}\right) \right] & \text{PC1} \\ \bar{\rho}_{sh} \left\{ 1 - \bar{e}_{2sh} \left[1 - \cos\left(\frac{\pi z}{h}\right) \right] \right\} & \text{PC2} \\ \bar{\rho}_{sh} \bar{e}_{3sh} & \text{PC3} \end{cases} \quad (2)$$

$$\bar{E}_{pc}^{sh} = \begin{cases} \bar{E}_{sh} \left[1 - e_{1sh} \cos\left(\frac{\pi z}{h}\right) \right] & \text{PC1} \\ \bar{E}_{sh} \left\{ 1 - e_{2sh} \left[1 - \cos\left(\frac{\pi z}{h}\right) \right] \right\} & \text{PC2} \\ \bar{E}_{sh} e_{3sh} & \text{PC3} \end{cases} \quad (3)$$

and for stiffeners $(h/2 \leq z \leq h/2 + h_{st})$

$$\bar{\nu}_{pc}^{st} = \bar{\nu}_{st}, \bar{\alpha}_{pc}^{st} = \bar{\alpha}_{st}, \quad (4)$$

$$\bar{\rho}_{pc}^{st} = \begin{cases} \bar{\rho}_{st} \left[1 - \bar{e}_{1st} \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right] & \text{PC1} \\ \bar{\rho}_{st} \left\{ 1 - \bar{e}_{2st} \left[1 - \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right] \right\} & \text{PC2} \\ \bar{\rho}_{st} \bar{e}_{3st} & \text{PC3} \end{cases} \quad (5)$$

$$\bar{E}_{pc}^{st} = \begin{cases} \bar{E}_{st} \left[1 - e_{1st} \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right] & \text{PC1} \\ \bar{E}_{st} \left\{ 1 - e_{2st} \left[1 - \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right] \right\} & \text{PC2} \\ \bar{E}_{st} e_{3st} & \text{PC3} \end{cases} \quad (6)$$

where e_{1sh} , e_{2sh} , e_{3sh} and e_{1st} , e_{2st} , e_{3st} denote the porosity coefficients of the shell and the stiffeners, respectively. The parameters $\bar{e}_{1sh}$, $\bar{e}_{2sh}$, $\bar{e}_{3sh}$ and $\bar{e}_{1st}$, $\bar{e}_{2st}$, $\bar{e}_{3st}$ represent the corresponding mass density coefficients. The quantities $\bar{\nu}_{sh}$, $\bar{\alpha}_{sh}$, $\bar{\rho}_{sh}$, $\bar{E}_{sh}$ and $\bar{\nu}_{st}$, $\bar{\alpha}_{st}$, $\bar{\rho}_{st}$, $\bar{E}_{st}$ refer to the Poisson's ratio, thermal expansion coefficient, mass density, and Young's modulus of the shell and stiffeners without porosity, respectively.

The GPL volume fractions in the shell and the stiffeners are given by

$$\hat{V}_G^{sh} = \frac{\bar{\rho}_m W_{GP}}{\bar{\rho}_m W_{GP} + \bar{\rho}_{GPL} (1 - W_{GP})}, \quad (7)$$

$$\hat{V}_G^{st} = \frac{\bar{\rho}_m W_{GP}}{\bar{\rho}_m W_{GP} + \bar{\rho}_{GPL} (1 - W_{GP})}, \quad (8)$$

where the subscripts G and m denote the graphene platelets and matrix, respectively, while $\bar{\rho}$ and W represent the density and mass fraction.

The effective elastic modulus is evaluated using the Halpin–Tsai model as follows

$$\bar{E}_{sh} = \frac{\bar{E}_m}{8} \left[\frac{3(1 + \bar{m}_{11}\bar{n}_{11}\hat{V}_G^{sh})}{1 - \bar{n}_{11}\hat{V}_G^{sh}} + \frac{5(1 + \bar{m}_{22}\bar{n}_{22}\hat{V}_G^{sh})}{1 - \bar{n}_{22}\hat{V}_G^{sh}} \right], \quad (9)$$

$$\bar{E}_{st} = \frac{\bar{E}_m}{8} \left[\frac{3(1 + \bar{m}_{11}\bar{n}_{11}\bar{V}_G^{st})}{1 - \bar{n}_{11}\bar{V}_G^{st}} + \frac{5(1 + \bar{m}_{22}\bar{n}_{22}\bar{V}_G^{st})}{1 - \bar{n}_{22}\bar{V}_G^{st}} \right], \quad (10)$$

where

$$\bar{n}_{11} = \frac{\bar{E}_{GPL} - \bar{E}_m}{\bar{E}_{GPL} + \bar{m}_{11}\bar{E}_m}, \quad \bar{n}_{22} = \frac{\bar{E}_{GPL} - \bar{E}_m}{\bar{E}_{GPL} + \bar{m}_{22}\bar{E}_m},$$

$$\bar{m}_{11} = \frac{2\bar{a}_G}{\bar{t}_G}, \quad \bar{m}_{22} = \frac{2\bar{b}_G}{\bar{t}_G},$$

where $\bar{E}_m$ and $\bar{E}_{GPL}$ denote the elastic moduli of the matrix and GPLs, respectively, and $\bar{a}_G$, $\bar{b}_G$, and $\bar{t}_G$ represent the length, width, and thickness of the GPLs.

The Poisson's ratio, thermal expansion coefficient, and mass density of the shell and stiffeners are evaluated using the rule of mixtures as follows

$$\begin{aligned} \bar{v}_{sh} &= \bar{v}_m(1 - \hat{V}_G^{sh}) + \bar{v}_G \hat{V}_G^{sh}, \\ \bar{v}_{st} &= \bar{v}_m(1 - \hat{V}_G^{st}) + \bar{v}_G \hat{V}_G^{st}, \\ \bar{\alpha}_{sh} &= \bar{\alpha}_m(1 - \hat{V}_G^{sh}) + \bar{\alpha}_G \hat{V}_G^{sh}, \\ \bar{\alpha}_{st} &= \bar{\alpha}_m(1 - \hat{V}_G^{st}) + \bar{\alpha}_G \hat{V}_G^{st}, \\ \bar{\rho}_{sh} &= \bar{\rho}_m(1 - \hat{V}_G^{sh}) + \bar{\rho}_G \hat{V}_G^{sh}, \\ \bar{\rho}_{st} &= \bar{\rho}_m(1 - \hat{V}_G^{st}) + \bar{\rho}_G \hat{V}_G^{st}. \end{aligned} \quad (11)$$

The relationship between the elastic modulus and mass density of the FG-GPLRPC shell can be expressed as follows [33]

$$\frac{\bar{E}_{pc}^{sh}}{\bar{E}_{sh}} = \left[\frac{\bar{\rho}_{pc}^{sh}}{\bar{\rho}_{sh}} \right]^2. \quad (12)$$

and for the stiffeners

$$\frac{\bar{E}_{pc}^{st}}{\bar{E}_{st}} = \left[\frac{\bar{\rho}_{pc}^{st}}{\bar{\rho}_{st}} \right]^2. \quad (13)$$

By substituting Eqs. (2), (3), (5), and (6) into Eqs. (12) and (13), the relations can be extended to FG-GPLRPC structures for the shell as follows

$$\begin{cases} 1 - \bar{e}_{1sh} \cos\left(\frac{\pi Z}{h}\right) = \sqrt{1 - e_{1sh} \cos\left(\frac{\pi Z}{h}\right)} \\ 1 - \bar{e}_{2sh} \left[1 - \cos\left(\frac{\pi Z}{h}\right)\right] = \sqrt{1 - e_{2sh} \left[1 - \cos\left(\frac{\pi Z}{h}\right)\right]} \\ \bar{e}_{3sh} = \sqrt{e_{3sh}} \end{cases} \quad (14)$$

and for the stiffeners

$$\begin{cases} 1 - \bar{e}_{1st} \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) = \sqrt{1 - e_{1st} \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right)} \\ 1 - \bar{e}_{2st} \left[1 - \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right] = \sqrt{1 - e_{2st} \left[1 - \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right]} \\ \bar{e}_{3st} = \sqrt{e_{3st}} \end{cases} \quad (15)$$

Assuming that the mass of the metal foam matrix remains constant for different porosity distribution patterns, the following relations for the shell can be obtained as

$$\begin{aligned} \int_0^{\frac{h}{2}} \sqrt{1 - e_{1sh} \cos\left(\frac{\pi Z}{h}\right)} dz \\ = \int_0^{\frac{h}{2}} \sqrt{1 - e_{2sh} \left[1 - \cos\left(\frac{\pi Z}{h}\right)\right]} dz = \int_0^{\frac{h}{2}} \sqrt{e_{3sh}} dz \end{aligned} \quad (16)$$

and for the stiffeners

$$\begin{aligned} \int_{\frac{h}{2} + h_{st}}^{\frac{h}{2}} \sqrt{1 - e_{1st} \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right)} dz = \int_{\frac{h}{2} + h_{st}}^{\frac{h}{2}} \sqrt{e_{3st}} dz \\ = \int_{\frac{h}{2} + h_{st}}^{\frac{h}{2}} \sqrt{1 - e_{2st} \left[1 - \cos \pi \left(\frac{2z - h - h_{st}}{2h_{st}} \right) \right]} dz, \end{aligned} \quad (17)$$

3. Nonlinear Vibration Formulation

Based on Donnell shell theory incorporating von Kármán geometric nonlinearity, the strain components of the conical shell at a distance from the mid-surface can be expressed as [3]

$$\varepsilon_x = z\chi_x + \varepsilon_x^0, \quad \varepsilon_\theta = z\chi_\theta + \varepsilon_\theta^0, \quad \gamma_{x\theta} = z\chi_{x\theta} + \gamma_{x\theta}^0, \quad (18)$$

where

$$\varepsilon_x^0 = u_{,x} + \frac{w_{,x}^2}{2},$$

$$\varepsilon_\theta^0 = \frac{v_{,\theta}}{x \sin(\delta)} + \frac{u}{x} + \frac{w \cot(\delta)}{x} + \frac{w_{,\theta}^2}{2x^2 \sin^2(\delta)}, \quad (19)$$

$$\gamma_{x\theta}^0 = \frac{u_{,\theta}}{x \sin(\delta)} - \frac{v}{x} + v_{,x} + \frac{w_{,x} w_{,\theta}}{x \sin(\delta)},$$

$$\chi_x = -w_{,xx}, \quad \chi_\theta = -\frac{w_{,\theta\theta}}{x^2 \sin^2(\delta)} - \frac{w_{,x}}{x}, \quad (20)$$

$$\chi_{x\theta} = -\frac{2w_{,x\theta}}{x \sin(\delta)} + \frac{2w_{,\theta}}{x^2 \sin(\delta)}.$$

The constitutive relations for FG-GPLRPC conical shells, incorporating thermal strain effects, are based on Hooke's law and can be expressed as

$$\sigma_x = Q_{11}[\varepsilon_x - \Delta T \bar{\alpha}(z)] + Q_{12}[\varepsilon_\theta - \Delta T \bar{\alpha}(z)],$$

$$\sigma_\theta = Q_{12}[\varepsilon_x - \Delta T \bar{\alpha}(z)] + Q_{22}[\varepsilon_\theta - \Delta T \bar{\alpha}(z)], \quad (21)$$

$$\tau_{x\theta} = Q_{66} \gamma_{x\theta},$$

where ΔT denotes the uniform temperature change, and the reduced stiffness coefficients of the shell are defined as

$$Q_{11}^{sh} = Q_{22}^{sh} = \frac{\bar{E}_{pc}^{sh}}{1 - (\bar{\nu}_{pc}^{sh})^2},$$

$$Q_{12}^{sh} = \frac{\bar{E}_{pc}^{sh} \bar{\nu}_{pc}^{sh}}{1 - (\bar{\nu}_{pc}^{sh})^2}, \quad Q_{66}^{sh} = \frac{\bar{E}_{pc}^{sh}}{2(1 + \bar{\nu}_{pc}^{sh})}, \quad (22)$$

for stiffeners

$$Q_{22}^{st} = \bar{E}_{pc}^{st}, \quad Q_{11}^{st} = Q_{12}^{st} = Q_{66}^{st} = 0, \quad (23)$$

The resultant internal forces and moments of the FG-GPLRPC conical shell are obtained by integrating the constitutive relations through the shell thickness. The contribution of the ring stiffeners is incorporated using an improved

Lekhnitskii smeared stiffener technique based on classical beam theory, accounting for both mechanical and thermal effects. Accordingly, the forces and moments of the stiffened shell can be expressed as follows

$$N_x = \hat{A}_{11} \varepsilon_x^0 + \hat{A}_{12} \varepsilon_\theta^0 + \hat{B}_{11} \chi_x + \hat{B}_{12} \chi_\theta - \Phi_{1x} \Delta T,$$

$$N_\theta = \hat{A}_{12} \varepsilon_x^0 + \hat{A}_{22} \varepsilon_\theta^0 + \hat{B}_{12} \chi_x + \hat{B}_{22} \chi_\theta - \Phi_{1\theta} \Delta T, \quad (24)$$

$$N_{x\theta} = \hat{A}_{66} \gamma_{x\theta}^0 + \hat{B}_{66} \chi_{x\theta},$$

$$M_x = \hat{B}_{11} \varepsilon_x^0 + \hat{B}_{12} \varepsilon_\theta^0 + \hat{D}_{11} \chi_x + \hat{D}_{12} \chi_\theta - \Phi_{2x} \Delta T,$$

$$M_\theta = \hat{B}_{12} \varepsilon_x^0 + \hat{B}_{22} \varepsilon_\theta^0 + \hat{D}_{12} \chi_x + \hat{D}_{22} \chi_\theta - \Phi_{2\theta} \Delta T, \quad (25)$$

$$M_{x\theta} = \hat{B}_{66} \gamma_{x\theta}^0 + \hat{D}_{66} \chi_{x\theta},$$

where the stiffnesses are expressed as

$$(\hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{sh}(1, z, z^2) dz, \quad (i, j = 1, 6)$$

$$(\hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{sh}(1, z, z^2) dz$$

$$+ \frac{b_{st}}{d_{st}} \int_{h/2}^{h/2+h_{st}} Q_{ij}^{st}(1, z, z^2) dz, \quad (i, j = 2) \quad (26)$$

$$(\Phi_{1x}, \Phi_{2x}) = \int_{-h/2}^{h/2} (Q_{11}^{sh} \bar{\alpha}_{sh} + Q_{12}^{sh} \bar{\alpha}_{sh})(1, z) dz,$$

$$(\Phi_{1\theta}, \Phi_{2\theta}) = \int_{-h/2}^{h/2} (Q_{12}^{sh} \bar{\alpha}_{sh} + Q_{22}^{sh} \bar{\alpha}_{sh})(1, z) dz$$

$$+ \frac{b_{st}}{d_{st}} \int_{h/2}^{h/2+h_{st}} (Q_{12}^{st} \bar{\alpha}_{st} + Q_{22}^{st} \bar{\alpha}_{st})(1, z) dz,$$

The thermo-elastic strain energy and kinetic energy of the FG-GPLRPC conical shell can be expressed as

$$\hat{U}_{int} = \frac{1}{2} \int_{x_0}^{x_0+L} \int_0^{2\pi} \int_{-h/2}^{h/2} (\sigma_x [\varepsilon_x - \bar{\alpha}(z) \Delta T]$$

$$+ \sigma_\theta [\varepsilon_\theta - \bar{\alpha}(z) \Delta T] + \tau_{x\theta} \gamma_{x\theta}) x \sin(\delta) dz d\theta dx \quad (27)$$

$$\hat{U}_t = \frac{1}{2} \int_{x_0}^{x_0+L} \int_0^{2\pi} \int_{-h/2}^{h/2} \bar{\rho}(z) w_{,t}^2 x \sin(\delta) dz d\theta dx. \quad (28)$$

In this study, the nonlinear free vibration of FG-GPLRPC conical shells is analyzed under simply supported and immovable boundary conditions at both edges. The considered boundary conditions are given as

$$\text{at } x = x_0, x = L + x_0 : M_x = 0, u = v = \dot{u} = 0. \quad (29)$$

The displacement field and transverse deflection are approximated by admissible functions that satisfy the boundary conditions in Eq. (29) as well as the circumferential continuity of the shell, given by [23]

$$\begin{aligned} u &= U \sin \left[\frac{2m\pi(x - x_0)}{L} \right] \sin(n\theta), \\ v &= V \sin \left[\frac{m\pi(x - x_0)}{L} \right] \sin(2n\theta), \\ w &= W \sin \left[\frac{m\pi(x - x_0)}{L} \right] \sin(n\theta). \end{aligned} \quad (30)$$

The total potential energy of the system can be expressed as

$$\hat{U}_{\text{total}} = U_t - U_{\text{int}}. \quad (31)$$

The damping potential is introduced through the Rayleigh dissipation function and can be expressed as

$$d_c = \frac{1}{2} \int_{x_0}^{x_0+L} \int_0^{2\pi} c w_{,t}^2 x \sin(\delta) d\theta dx. \quad (32)$$

By incorporating the Rayleigh dissipation function into the Euler–Lagrange equations, the governing equations of motion can be derived as

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \hat{U}_{\text{total}}}{\partial \dot{U}} \right) - \frac{\partial \hat{U}_{\text{total}}}{\partial U} &= 0, \\ \frac{d}{dt} \left(\frac{\partial \hat{U}_{\text{total}}}{\partial \dot{V}} \right) - \frac{\partial \hat{U}_{\text{total}}}{\partial V} &= 0, \\ \frac{d}{dt} \left(\frac{\partial \hat{U}_{\text{total}}}{\partial \dot{W}} \right) - \frac{\partial \hat{U}_{\text{total}}}{\partial W} + \frac{\partial d_c}{\partial \dot{W}} &= 0, \end{aligned} \quad (33)$$

leads to

$$U \bar{b}_1 + W \bar{b}_2 = 0, \quad (34)$$

$$W^2 \bar{b}_3 + V \bar{b}_4 = 0, \quad (35)$$

$$\begin{aligned} W^3 \bar{b}_5 + V W \bar{b}_6 + W \bar{b}_7 \Delta T + U \bar{b}_8 + W \bar{b}_9 \\ + \dot{W} c \bar{b}_{10} + \ddot{W} \bar{b}_{11} = 0, \end{aligned} \quad (36)$$

The in-plane displacement amplitudes are first determined from Eqs. (34) and (35). Substituting these expressions into Eq. (36) yields

the governing equation of motion as

$$\begin{aligned} W^3 \left(\bar{b}_5 - \frac{\bar{b}_3 \bar{b}_6}{\bar{b}_4} \right) + W \left(\bar{b}_9 - \frac{\bar{b}_2 \bar{b}_8}{\bar{b}_1} \right) \\ + W \bar{b}_7 \Delta T + \dot{W} c \bar{b}_{10} + \ddot{W} \bar{b}_{11} = 0, \end{aligned} \quad (37)$$

where the coefficients $\bar{b}_i$ are expressed in the Appendix.

The governing equation for linear free vibration is obtained by neglecting the nonlinear terms. Accordingly, the natural frequency of linear free vibration can be expressed as

$$\omega_{mn} = \sqrt{\frac{\bar{b}_7 \Delta T + \bar{b}_9 - \frac{\bar{b}_2 \bar{b}_8}{\bar{b}_1}}{\bar{b}_{11}}}. \quad (38)$$

The vibration amplitude is assumed in the harmonic form as

$$W = \Xi_1 \sin(\Omega t) + \Xi_2 \cos(\Omega t) \quad (39)$$

Substituting the assumed solution into Eq. (37) and applying the harmonic balance method, while neglecting the damping and higher-order terms under the assumption of weak nonlinearity and small damping, the following expressions are obtained

$$-\bar{K}_5 \Xi_1 \Omega^2 + \bar{K}_1 \Xi_1 + \frac{3}{4} \bar{K}_3 \Xi_1^3 + \frac{3}{2} \bar{K}_3 \Xi_1 \Xi_2^2 = 0 \quad (40)$$

$$-\bar{K}_5 \Xi_2 \Omega^2 + \bar{K}_1 \Xi_2 + \frac{3}{2} \bar{K}_3 \Xi_1^2 \Xi_2 + \frac{3}{4} \bar{K}_3 \Xi_2^3 = 0 \quad (41)$$

where

$$\bar{K}_1 = \bar{b}_9 - \frac{\bar{b}_2 \bar{b}_8}{\bar{b}_1} + \bar{b}_7 \Delta T,$$

$$\bar{K}_3 = \bar{b}_5 - \frac{\bar{b}_3 \bar{b}_6}{\bar{b}_4}, \quad \bar{K}_5 = \bar{b}_{11}.$$

By introducing the polar coordinate transformation $\Xi_1 = \Xi \sin(\psi)$, $\Xi_2 = \Xi \cos(\psi)$ and substituting it into Eqs. (40) and (41), the following relations are obtained:

$$\begin{aligned} \Xi \sin(\psi) \left(\bar{K}_1 - \bar{K}_5 \Omega^2 + \frac{3}{4} \bar{K}_3 \Xi^2 \right) \\ + \frac{3}{4} \bar{K}_3 \Xi^3 \sin(\psi) \cos^2(\psi) = 0 \end{aligned} \quad (42)$$

$$\Xi \cos(\psi) \left(\bar{K}_1 - \bar{K}_5 \Omega^2 + \frac{3}{4} \bar{K}_3 \Xi^2 \right) + \frac{3}{4} \bar{K}_3 \Xi^3 \sin^2(\psi) \cos(\psi) = 0 \quad (43)$$

Solving Eqs. (42) and (43) yields the frequency–amplitude relationship Ω – Ξ in the following approximate form

$$\Omega = \frac{1}{2} \sqrt{\frac{2}{K_5^2} \left(2K_1 K_5 + \frac{15K_3 K_5 \Xi^2}{8} \right)} \quad (44)$$

4. Results and discussions

Table 1 presents a comparison of the dimensionless natural frequencies of isotropic conical shells obtained in the present study with those reported in previous works. It can be observed that the predicted frequency value shows good agreement with the results of Irie et al. [2] and Lam and Hua [1], thereby validating the accuracy of the present formulation. .

Table 1. Comparison of dimensionless natural frequencies $\hat{\omega} = \omega_{mn} R_2 \sqrt{(1 - \bar{v}_m^2) \bar{\rho}_m / \bar{E}_m}$ of isotropic conical shells with previous studies ($m = 1$, $n = 2$, $\bar{v}_m = 0.3$, $h/R_2 = 0.01$, $\delta = 60^\circ$, $L \sin \delta / R_2 = 0.25$).

Irie et al. [2]	0.5722
Lam and Hua [1]	0.6348
Present	0.6702

Table 2. Fundamental frequencies of FG-GPLRPC conical shells for different GPL mass fractions, porosity distributions, and stiffener configurations (rad/s, $h = 0.01\text{m}$, $R_2 = 100h$, $H = 2R_2$, $\delta = 15^\circ$,

$$h_{st} = 1.5h, b_{st} = h, d_{st} = L/4, e_{1sh} = e_{1st} = 0.3, \Delta T = 100\text{K})$$

	W_{GP}	PC1/PC1	PC2/PC2	PC3/PC3
Unstiffened	0%	4918.12 (1;8)	4869.48 (1;9)	4889.26 (1;8)
	3%	5373.77 (1;8)	5323.32 (1;9)	5342.31 (1;8)
	6%	5793.34 (1;7)	5742.58 (1;9)	5760.40 (1;8)
	9%	6183.53 (1;7)	6134.28 (1;9)	6150.62 (1;8)
Ring stiffeners	0%	4987.46 (7;1)	4939.40 (8;1)	4961.11 (8;1)
	3%	5448.34 (7;1)	5398.57 (8;1)	5422.09 (8;1)
	6%	5873.42 (7;1)	5822.57 (8;1)	5845.19 (7;1)
	9%	6269.99 (7;1)	6218.53 (8;1)	6239.90 (7;1)

Table 2 presents the fundamental frequencies of FG-GPLRPC conical shells for different GPL mass fractions, porosity distributions, and stiffener configurations. It can be clearly observed that the natural frequency increases with the GPL mass fraction for all porosity patterns. This trend is attributed to the significant enhancement in the effective stiffness of the composite due to the superior mechanical properties of graphene platelets, which outweigh the corresponding increase in mass density. The influence of porosity distribution is also evident. Among the three

patterns, the PC1 configuration generally yields the highest frequencies, followed by PC3 and PC2. This behavior can be explained by the distribution of pores across the thickness: when larger pores are concentrated near the mid-surface (PC1), the reduction in bending stiffness is less pronounced compared to cases where pores are more influential near regions contributing significantly to structural rigidity. In contrast, the PC2 distribution leads to relatively lower stiffness and hence lower frequencies. The ring stiffeners provide a moderate increase in natural frequency, as the stiffened

configurations exhibit higher natural frequencies compared to the unstiffened ones. The addition of stiffeners enhances the overall structural rigidity, particularly in the circumferential direction, thereby increasing resistance to deformation and improving dynamic performance. Furthermore, a noticeable change in the dominant vibration modes is observed between unstiffened and stiffened shells. This indicates that the presence of stiffeners not only alters the stiffness characteristics but also modifies the modal behavior of the structure.

Table 3 presents the fundamental frequencies of FG-GPLRPC conical shells for different semi-vertex angles and porosity distributions. It is evident that the natural frequency increases significantly with the semi-vertex angle δ . Specifically, as δ increases from 10° to 20° , the frequency rises markedly for all porosity patterns. This behavior can be attributed to the geometric

stiffening effect, as larger cone angles lead to enhanced structural rigidity and reduced flexibility. The influence of porosity distribution is also clearly observed. In most cases, the PC1 configuration yields higher frequencies; however, slight deviations can be observed for certain geometric configurations, indicating a complex interaction between porosity distribution and shell geometry. Additionally, the corresponding mode shapes, indicated by the pairs $(m;n)$, exhibit variations depending on both the semi-vertex angle and porosity pattern. For smaller angles ($\delta = 10^\circ$ and 15°), the dominant modes are primarily characterized by higher axial wave numbers, whereas at $\delta = 20^\circ$, a transition in mode shapes is observed. This suggests that geometric parameters not only influence the stiffness but also alter the vibration characteristics and modal behavior of the shell.

Table 3. Fundamental frequencies of FG-GPLRPC conical shells for different semi-vertex angles and porosity distributions (rad/s, $h=0.01\text{m}$, $R_2=100h$, $H=2R_2$, $h_{st}=1.5h$, $b_{st}=h$, $d_{st}=L/4$, $e_{1st}=e_{1sh}$,

$$W_{GP}=5\%, \Delta T=100\text{K})$$

	e_{1sh}	PE-PC1/PC1	PE-PC2/PC2	PE-PC3/PC3
$\delta = 10^\circ$	0.2	5150.48 (7;1)	5108.22 (8;1)	5133.22 (7;1)
	0.4	4976.24 (7;1)	4895.90 (9;1)	4928.08 (7;1)
	0.6	4791.07 (7;1)	4678.71 (10;1)	4684.82 (7;1)
$\delta = 15^\circ$	0.2	5834.79 (7;1)	5798.40 (8;1)	5819.58 (7;1)
	0.4	5631.83 (7;1)	5568.57 (9;1)	5587.02 (7;1)
	0.6	5414.81 (7;1)	5347.96 (10;1)	5311.23 (7;1)
$\delta = 20^\circ$	0.2	6732.61 (1;3)	6736.94 (1;4)	6727.25 (1;4)
	0.4	6481.90 (1;3)	6520.26 (1;4)	6458.41 (1;4)
	0.6	6209.90 (1;3)	6356.63 (10;1)	6139.60 (1;4)

Fig. 2 illustrates the effects of ring stiffeners on the vibration response of FG-GPLRPC conical shells from several complementary perspectives, including the frequency–amplitude relationship, time-history response, phase-plane trajectory, and the influence of stiffener spacing. As shown in Fig. 2(a), the stiffened shell exhibits a lower vibration amplitude than the unstiffened one at the same

normalized frequency level. In both cases, the frequency–amplitude curves bend to the right, indicating a hardening-type nonlinear behavior. However, the stiffened shell follows a distinctly lower response path, demonstrating that the presence of ring stiffeners suppresses transverse deflection and improves vibration resistance. The time-history response in Fig. 2(b) further supports

this observation. The oscillation amplitude of the stiffened shell remains slightly smaller than that of the unstiffened shell, while the periodic nature of the response is preserved. This indicates that the stiffeners primarily reduce vibration amplitude rather than fundamentally altering the oscillatory pattern. A similar trend can be observed in the phase-plane plot shown in Fig. 2(c). The trajectory of the stiffened shell encloses a smaller region than that of the unstiffened shell, implying a reduced vibration amplitude and lower oscillation energy.

The nearly closed elliptical shape of both trajectories indicates stable periodic motion. Fig. 2(d) illustrates the effect of stiffener spacing on the frequency–amplitude response. As the spacing between ring stiffeners decreases, the shell response is further reduced, leading to lower vibration amplitudes at a given normalized frequency. This trend is physically reasonable, since a smaller stiffener spacing corresponds to a denser reinforcement layout and thus a greater overall structural stiffness.

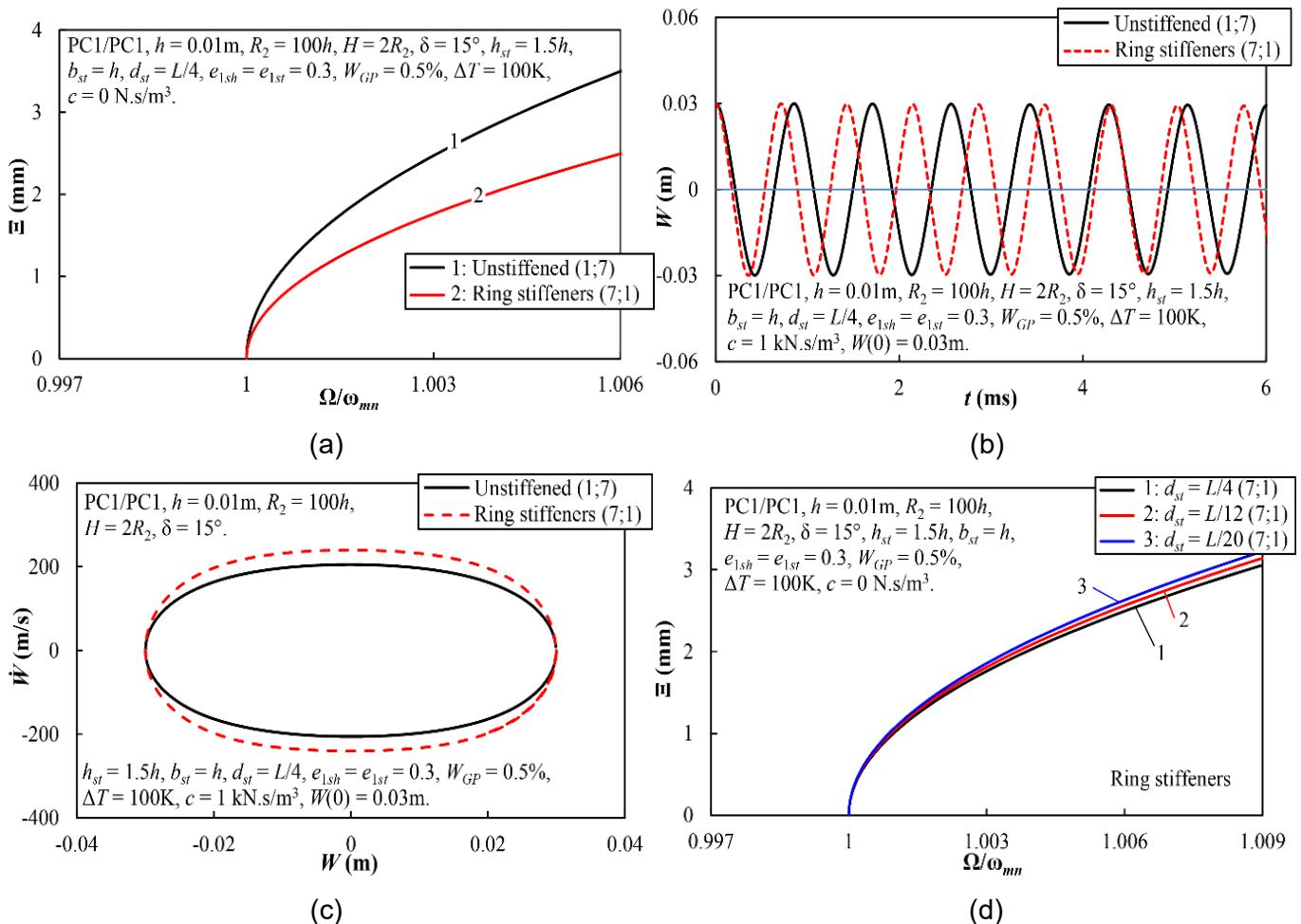


Fig. 2. Effects of stiffeners on the vibration responses of FG-GPLRPC stiffened conical shells. (a) Frequency–amplitude curve; (b) Time-history response; (c) Amplitude-velocity curve (phase plane); (d) Frequency–amplitude curve (different stiffener spacings)

Fig. 3 presents the effects of temperature and material parameters on the vibration response of FG-GPLRPC stiffened conical shells through frequency–amplitude curves, time-history responses, and phase-plane trajectories. As shown in Fig. 3(a), increasing the temperature

change from $\Delta T = 100\text{ K}$ to 200 K shifts the frequency–amplitude curve downward. For a given normalized frequency, the transverse vibration amplitude becomes smaller as the temperature increases. Equivalently, for a fixed amplitude level, a higher temperature corresponds to a lower

nonlinear vibration frequency. This trend indicates a thermal softening effect, since the temperature rise reduces the effective stiffness of the shell through thermoelastic action. Nevertheless, the

curves still bend to the right, confirming that the system continues to exhibit hardening-type nonlinearity despite the stiffness degradation induced by temperature.

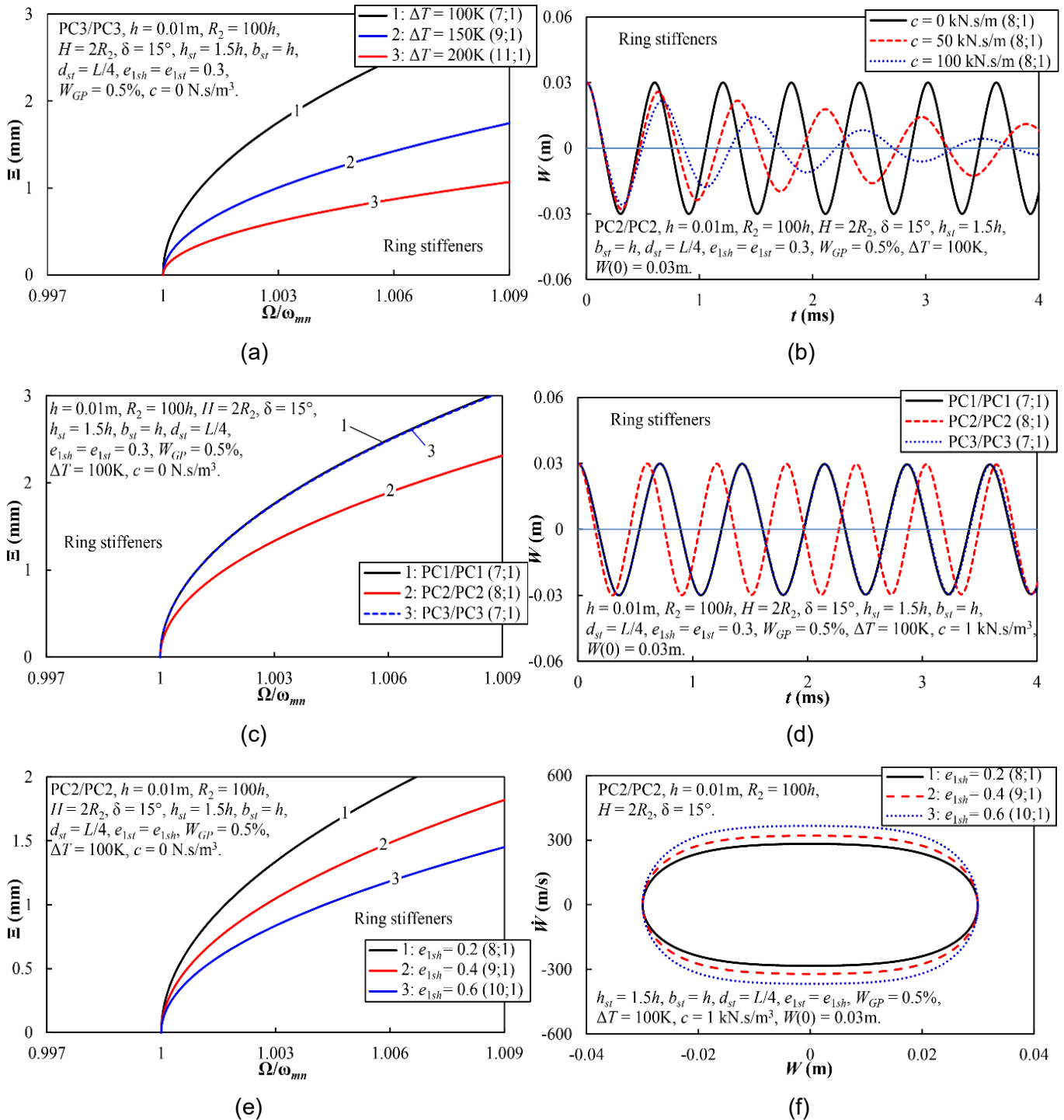


Fig. 3. Effects of temperature and material properties on the vibration responses of FG-GPLRPC stiffened conical shells. (a) Frequency–amplitude curve (different temperature changes); (b) Time-history response (different damping coefficients); (c) Frequency–amplitude curve (different porosity distributions); (d) Time-history response (different porosity distributions); (e) Frequency–amplitude curve (different porosity coefficients); (f) Amplitude-velocity curve (phase plane) (different porosity coefficients)

Fig. 3(b) shows that increasing the damping coefficient leads to a more rapid decay of vibration amplitude. While the undamped system exhibits sustained periodic motion, higher damping levels result in faster stabilization of the system, whereas the oscillation frequency remains only slightly affected.

Fig. 3(c) compares the influence of porosity distribution patterns. The PC1 and PC3 configurations produce higher frequency–amplitude curves than PC2, while PC2 gives the lowest response over the entire range. This indicates that the distribution of pores through the thickness plays an important role in determining the effective stiffness of the shell. A more favorable porosity arrangement preserves the load-carrying capability and bending rigidity more effectively, whereas a less efficient distribution leads to a more

pronounced stiffness reduction and, consequently, a lower vibration response.

The time-history responses in Fig. 3(d) support this conclusion. Although all three cases remain periodic and stable, differences in oscillation frequency are visible, with the less favorable porosity distribution showing a slightly slower oscillatory response.

The effect of the porosity coefficient is further illustrated in Fig. 3(e). As the porosity coefficient increases from $e_1=0.2$ to 0.6, the frequency–amplitude curve moves downward substantially. This behavior is physically expected because a larger porosity coefficient corresponds to a higher void content, which reduces the effective elastic modulus and weakens the structural stiffness. As a result, the shell becomes more flexible, and its vibration frequency decreases.

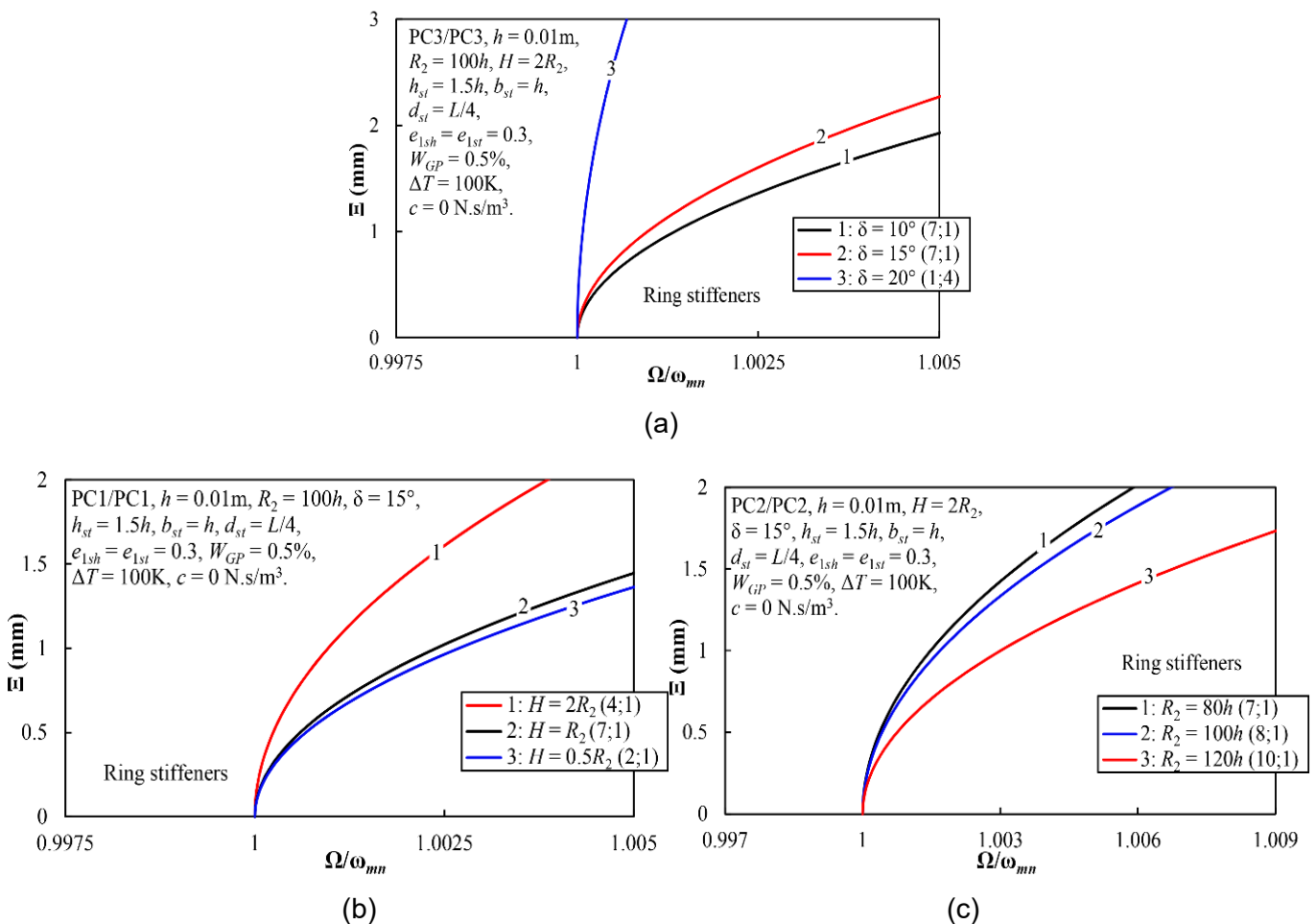


Fig. 4. Effects of geometrical properties on the frequency–amplitude responses of FG-GPLRPC stiffened conical shells (a) Different semi-vertex angles; (b) Different axial lengths; (c) Different radii of large base

The phase-plane plots in Fig. 3(f) are consistent with this interpretation. Higher porosity coefficients lead to larger closed trajectories, indicating stronger vibration response and greater oscillation amplitude in the displacement–velocity space. At the same time, the nearly elliptical closed loops suggest that the motion remains stable and periodic within the considered parameter range.

Fig. 4 illustrates the influence of geometric parameters on the nonlinear frequency–amplitude response of FG-GPLRPC conical shells reinforced with ring stiffeners. In all cases, the curves exhibit a rightward inclination, indicating a typical hardening-type nonlinear behavior. As shown in Fig. 4(a), increasing the semi-vertex angle δ from 10° to 20° shifts the frequency–amplitude curves upward. This implies that, for a given vibration amplitude, the nonlinear frequency increases with the cone angle. Such behavior can be attributed to the geometric stiffening effect, as larger cone angles enhance the structural rigidity and reduce the susceptibility to deformation. In addition, the variation in the associated vibration modes suggests that the semi-vertex angle not only affects stiffness but also alters the modal characteristics of the shell.

Fig. 4(b) presents the effect of the axial length H . It is observed that increasing the shell length leads to a downward shift of the frequency–amplitude curves, indicating a reduction in vibration frequency. This trend is physically consistent, since a longer shell exhibits greater flexibility and reduced bending stiffness, resulting in lower dynamic stiffness and hence lower natural frequencies.

The influence of the large base radius R_2 is depicted in Fig. 4(c). As R_2 increases from $80h$ to $120h$, the frequency–amplitude curves move downward, demonstrating a decrease in vibration frequency. This can be explained by the increase in structural slenderness and the associated reduction in effective stiffness as the radius

becomes larger.

5. Conclusion

An analytical investigation of the nonlinear free vibration response of FG-GPLRPC conical shells with circular FG-GPLRPC stiffeners in thermal environments is presented in this study. The formulation is established within the framework of Donnell shell theory combined with von Kármán geometric nonlinearity, while the influence of stiffeners is incorporated through an improved smeared stiffener technique. The governing equations were derived via the Ritz method and solved using the harmonic balance approach to obtain the nonlinear frequency–amplitude relationships. The numerical results demonstrated that the addition of ring stiffeners enhances structural stiffness and reduces vibration amplitudes, although its influence on natural frequency is relatively moderate. The influence of material parameters was found to be substantial, where increasing the graphene platelet content leads to higher natural frequencies due to stiffness enhancement, whereas higher porosity and elevated temperature reduce the vibration frequencies as a result of stiffness degradation. In addition, the porosity distribution pattern plays an important role in governing the effective stiffness, with more favorable distributions yielding improved dynamic performance.

The geometric parameters were also shown to have pronounced effects on the vibration characteristics. In particular, increasing the semi-vertex angle improves structural rigidity and raises the nonlinear frequency, whereas increasing the axial length and base radius reduces the stiffness and leads to lower frequencies. These results highlight the strong interplay between material design, structural reinforcement, and geometric configuration in determining the nonlinear dynamic response of FG-GPLRPC conical shells.

Declaration of Generative AI

During the preparation of this work, the

authors did not use Generative AI and AI-assisted technology tools in order to generate the content of the manuscript.

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Appendix

$$\bar{b}_1 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{\alpha_6^2 - 1}{xL^2\alpha_7^3} \left[\begin{aligned} & \left[(-4\hat{A}_{11}\pi^2m^2x^2 + \hat{A}_{22}L^2)\alpha_6^2 \right. \\ & \left. + (\hat{A}_{66}n^2 - \hat{A}_{22})L^2 + 4\hat{A}_{11}\pi^2m^2x^2 \right] \alpha_5^2\alpha_1^2 \\ & - (\alpha_6^2 - 1)(-4\hat{A}_{11}\pi^2m^2x^2 + \hat{A}_{22}L^2)\alpha_1^2 \\ & - 4Lm\pi\hat{A}_{12}(\alpha_5^2 - 1)(\alpha_6^2 - 1)\alpha_1\alpha_2 \\ & - L^2 \left[(\hat{A}_{22}\alpha_6^2 + \hat{A}_{66}n^2 - \hat{A}_{22})\alpha_5^2 - \hat{A}_{22}\alpha_6^2 + \hat{A}_{22} \right] \end{aligned} \right] \right\} d\theta dx$$

$$\bar{b}_2 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{\alpha_6^2 - 1}{x^2L^3\alpha_7^3} \left[\begin{aligned} & 2x\alpha_5 \left[\begin{aligned} & -\hat{B}_{11}\pi^2\alpha_6^2\alpha_4m^2x^2 + m^2\pi^2x^2\alpha_4\hat{B}_{11} \\ & + \hat{A}_{12}L^2\alpha_6\alpha_7\alpha_4x \\ & + L^2\hat{B}_{12}\alpha_4n^2 + L\hat{B}_{12}\alpha_3m\pi(\alpha_6^2 - 1) \end{aligned} \right] m\alpha_1\pi(\alpha_5 + 1) \\ & - \hat{B}_{12}L\pi^2\alpha_6^4\alpha_4\alpha_2m^2x^2 + \hat{B}_{12}L\pi^2m^2x^2\alpha_6^2\alpha_4\alpha_2 \\ & + \hat{A}_{22}L^3\alpha_6^3\alpha_7\alpha_4\alpha_2x + L^3n^2\alpha_6^2\alpha_4\alpha_2(\hat{B}_{22} - 2\hat{B}_{66}) \\ & + \hat{B}_{12}L\pi^2\alpha_6^2\alpha_4\alpha_2m^2x^2 - \hat{B}_{12}L\pi^2m^2x^2\alpha_4\alpha_2 \\ & - \hat{A}_{22}L^3\alpha_6\alpha_7\alpha_4\alpha_2x - \hat{B}_{22}L^3n^2\alpha_4\alpha_2 \\ & + \left[(\hat{B}_{22}\alpha_6^2 + 2n^2\hat{B}_{66} - \hat{B}_{22})\alpha_5^2 \right] xL^2m\pi\alpha_3\alpha_2 \\ & - \hat{B}_{22}\alpha_6^2 + \hat{B}_{22} \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_3 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ -\frac{n}{x^2L^2\alpha_7^2} \left[\begin{aligned} & L\alpha_5\alpha_8\alpha_9m\pi\hat{A}_{66}(\alpha_6^2 - 1)\alpha_3^3 \\ & \left[(-\hat{A}_{12}\pi^2\alpha_6^2m^2x^2 + \hat{A}_{12}\pi^2m^2x^2 + \hat{A}_{22}L^2n^2)\alpha_5^2\alpha_{10} \right. \\ & \left. + \hat{A}_{12}\pi^2m^2x^2(\alpha_6^2 - 1)\alpha_{10} \right. \\ & \left. + \alpha_5\alpha_8\alpha_9m^2x^2\pi^2\hat{A}_{66}(\alpha_6^2 - 1) \right] \alpha_4\alpha_3^2 \\ & - L\alpha_5\alpha_8\alpha_9m\pi\hat{A}_{66}(\alpha_6^2 - 1)\alpha_3 - L^2\hat{A}_{22}\alpha_5^2\alpha_{10}\alpha_4n^2 \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_4 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{\alpha_6^2 - 1}{xL^2\alpha_7^3} \left[\begin{aligned} & \left[\hat{A}_{66}(-m^2x^2\pi^2 + L^2)\alpha_6^2 + (m^2x^2\pi^2 - L^2)\hat{A}_{66} + 4A_{22}L^2n^2 \right] \alpha_{10}^2\alpha_3^2 \\ & - \hat{A}_{66}(\alpha_6^2 - 1)(L^2 - \pi^2m^2x^2)\alpha_3^2 \\ & + 2L\alpha_4m\pi\hat{A}_{66}(\alpha_{10}^2 - 1)(\alpha_6^2 - 1)\alpha_3 \\ & - L^2 \left[(4n^2\hat{A}_{22} + \hat{A}_{66}\alpha_6^2 - \hat{A}_{66})\alpha_{10}^2 - \hat{A}_{66}\alpha_6^2 + \hat{A}_{66} \right] \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_6 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{2n}{x^2L^2\alpha_7^2} \left[\begin{aligned} & -L\alpha_5\alpha_8\alpha_9m\pi\hat{A}_{66}(\alpha_6^2 - 1)\alpha_3^3 - \hat{A}_{12}\pi^2m^2x^2(\alpha_6^2 - 1)\alpha_{10}\alpha_4\alpha_3^2 \\ & - \left[-\hat{A}_{12}\pi^2m^2x^2(\alpha_6^2 - 1) + \hat{A}_{22}L^2n^2 \right] \alpha_5^2\alpha_{10}\alpha_4\alpha_3^2 \\ & - \alpha_5\alpha_8\alpha_9m^2x^2\pi^2\hat{A}_{66}(\alpha_6^2 - 1)\alpha_4\alpha_3^2 \\ & + L\alpha_5\alpha_8\alpha_9m\pi\hat{A}_{66}(\alpha_6^2 - 1)\alpha_3 + L^2\hat{A}_{22}\alpha_5^2\alpha_{10}\alpha_4n^2 \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_7 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ -\frac{\alpha_6^2 - 1}{xL^2\alpha_7^3} \left[\begin{aligned} & -m^2x^2\pi^2\Phi_{1x}(\alpha_6^2 - 1)\alpha_6^2\alpha_3^2 + \Phi_{10}n^2L^2\alpha_6^2\alpha_3^2 \\ & + m^2x^2\pi^2(\alpha_6^2 - 1)\Phi_{1x}\alpha_3^2 - L^2\alpha_5^2n^2\Phi_{10} \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_5 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{1}{2x^3 L^4 \alpha_7^3} \left[\begin{aligned} &\hat{A}_{11} \alpha_6^4 \alpha_5^4 \alpha_3^4 m^4 x^4 \pi^4 - 2m^4 x^4 \pi^4 \hat{A}_{11} \alpha_6^2 \alpha_5^4 \alpha_3^4 \\ &- (4\hat{A}_{66} + 2\hat{A}_{12}) L^2 n^2 m^2 x^2 \pi^2 \alpha_6^2 \alpha_5^4 \alpha_3^4 + \hat{A}_{11} \alpha_5^4 \alpha_3^4 m^4 x^4 \pi^4 \\ &+ L^2 m^2 n^2 \pi^2 \alpha_5^4 \alpha_3^4 (4\hat{A}_{66} + 2\hat{A}_{12}) x^2 + n^4 \hat{A}_{22} \alpha_5^4 \alpha_3^4 L^4 \\ &- 2\alpha_6^2 m^4 x^4 \pi^4 \hat{A}_{11} (\alpha_6^2 - 1) \alpha_5^2 \alpha_3^4 + 2m^4 x^4 \pi^4 \hat{A}_{11} (\alpha_6^2 - 1) \alpha_5^2 \alpha_3^4 \\ &+ L^2 n^2 m^2 x^2 \pi^2 \alpha_5^2 \alpha_3^4 (4\hat{A}_{66} + 2\hat{A}_{12}) (\alpha_6^2 - 1) \\ &+ \hat{A}_{11} \alpha_3^4 m^4 x^4 \pi^4 (\alpha_6^2 - 1)^2 - 2\hat{A}_{22} \alpha_5^4 L^4 n^4 \alpha_3^2 + \hat{A}_{22} L^4 \alpha_5^4 n^4 \\ &- 2(\alpha_6^2 - 1)(2\hat{A}_{66} + \hat{A}_{12}) m^2 x^2 \pi^2 \alpha_5^2 L^2 n^2 \alpha_3^2 \\ &+ 2(2\hat{A}_{66} + \hat{A}_{12}) (\alpha_6^2 - 1) m^2 x^2 \pi^2 \alpha_5^4 L^2 n^2 \alpha_3^2 \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_8 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{\alpha_6^2 - 1}{x^2 L^3 \alpha_7^3} \left[\begin{aligned} &-\hat{B}_{12} \pi^2 m^2 x^2 (\alpha_6^2 - 1) (\alpha_5^2 - 1) L \alpha_2 + \hat{A}_{22} L^3 \alpha_6 \alpha_7 \alpha_4 \alpha_2 x (\alpha_5^2 - 1) \\ &+ (\hat{B}_{22} - 2\hat{B}_{66}) L^3 n^2 \alpha_5^2 \alpha_4 \alpha_2 - 2\hat{B}_{11} \pi^3 m^3 x^3 (\alpha_5^2 - 1)^2 \alpha_4 \alpha_1 \\ &+ (\hat{B}_{22} \alpha_6^2 + 2n^2 \hat{B}_{66} - \hat{B}_{22}) \alpha_5^2 x L^2 m \pi \alpha_3 \alpha_2 \\ &- \hat{B}_{22} \alpha_5^2 x L^2 m \pi \alpha_3 \alpha_2 + \hat{B}_{22} x L^2 m \pi \alpha_3 \alpha_2 - \hat{B}_{22} \alpha_2 \alpha_4 L^3 n^2 \\ &+ 2\hat{A}_{12} L^2 \alpha_6 \alpha_7 \alpha_4 \alpha_1 m \pi x^2 (\alpha_5^2 - 1) + 2L^2 \hat{B}_{12} m n^2 \alpha_4 \alpha_1 \pi x (\alpha_5^2 - 1) \\ &+ 2L \hat{B}_{12} \alpha_3 \alpha_1 m^2 x^2 \pi^2 (\alpha_6^2 - 1) (\alpha_5^2 - 1) \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_9 = \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{1}{x^3 \alpha_7^3 L^4} \left[\begin{aligned} &-2(\alpha_6^2 - 1)(\alpha_5^2 - 1) \alpha_4 m x L \pi \alpha_3 \left[\begin{aligned} &L^2 n^2 \hat{D}_{22} - m^2 x^2 \pi^2 \hat{D}_{12} (\alpha_6^2 - 1) \\ &-\hat{B}_{22} L^2 \alpha_6 \alpha_7 x \end{aligned} \right] \\ &+ \left[\begin{aligned} &\hat{D}_{11} \pi^4 m^4 x^4 (\alpha_6^4 + 1) + \hat{A}_{22} L^4 x^2 \alpha_6^2 (\alpha_6^2 - 1) - L^4 n^4 \hat{D}_{22} \\ &+ 2L^2 \hat{B}_{12} \pi^2 \alpha_6 \alpha_7 m^2 x^3 (\alpha_6^2 - 1) + 2\hat{D}_{11} \pi^4 m^4 x^4 \alpha_6^2 \\ &+ 2\hat{D}_{12} \pi^2 m^2 n^2 L^2 x^2 \alpha_6^2 (\alpha_6^2 - 1) - 2\hat{B}_{22} L^4 n^2 \alpha_6 \alpha_7 x \end{aligned} \right] (\alpha_5^2 - 1) \\ &- 4\hat{D}_{66} L^3 n^2 \alpha_5^2 (\alpha_6^2 - 1) (L - 2\alpha_3 \alpha_4 m x \pi) \end{aligned} \right] \right\} d\theta dx,$$

$$+ \int_{x_0}^{x_0+L} \int_0^{2\pi} \left\{ \frac{1}{x^3 \alpha_7^3 L^4} \left[\begin{aligned} &\left[\begin{aligned} &\hat{D}_{11} \pi^4 m^4 x^4 (\alpha_6^4 + 1) - \hat{D}_{22} L^2 \pi^2 m^2 x^2 (\alpha_6^4 + 1) \\ &-\hat{A}_{22} L^4 x^2 \alpha_6^2 (\alpha_6^2 - 1) - 2\hat{B}_{12} L^2 \pi^2 \alpha_6 \alpha_7 m^2 x^3 (\alpha_6^2 - 1) \end{aligned} \right] (\alpha_5^2 - 1) \alpha_3^2 \\ &- 2\hat{D}_{11} \pi^4 m^4 x^4 \alpha_6^2 - 2\hat{D}_{12} n^2 m^2 \pi^2 L^2 x^2 (\alpha_6^2 - 1) \\ &+ 2\hat{D}_{22} m^2 \pi^2 L^2 x^2 \alpha_6^2 + 2\hat{B}_{22} L^4 n^2 \alpha_6 \alpha_7 x + \hat{D}_{22} L^4 n^4 \\ &+ (4\hat{D}_{66} L^4 n^2 - 4\hat{D}_{66} n^2 m^2 \pi^2 L^2 x^2) \alpha_5^2 (\alpha_6^2 - 1) \alpha_3^2 \end{aligned} \right] \right\} d\theta dx,$$

$$\bar{b}_{10} = \int_{x_0}^{x_0+L} \int_0^{2\pi} (\alpha_4^2 \alpha_8^2 \alpha_7 x) d\theta dx, \quad \bar{b}_{11} = \int_{x_0}^{x_0+L} \int_0^{2\pi} (\alpha_4^2 \alpha_8^2 \alpha_7 x l_1) d\theta dx,$$

$$l_1 = \int_{-h/2}^{h/2} \bar{\rho}_{sh} dz + \frac{b_{st}}{d_{st}} \int_{h/2}^{h/2+h_{st}} \bar{\rho}_{st} dz,$$

$$\alpha_1 = \cos\left(\frac{2m\pi(x-x_0)}{L}\right), \alpha_2 = \sin\left(\frac{2m\pi(x-x_0)}{L}\right),$$

$$\alpha_3 = \cos\left(\frac{m\pi(x-x_0)}{L}\right), \alpha_4 = \sin\left(\frac{m\pi(x-x_0)}{L}\right),$$

$$\alpha_5 = \cos(n\theta), \alpha_6 = \cos(\delta), \alpha_7 = \sin(\delta), \alpha_8 = \sin(n\theta), \alpha_9 = \sin(2n\theta), \alpha_{10} = \cos(2n\theta)$$