



Article info

Type of article:

Original research paper

DOI:

<https://doi.org/10.58845/jstt.utt.2026.en.6.3.380-397>

*Corresponding author:

Email address:

thanhhoai.nguyen@utc.edu.vn

Received: 29/03/2026

Received in Revised Form:
24/06/2026

Accepted: 18/07/2026

Impulsive responses of composite spherical shells reinforced by graphene platelets with spiderweb stiffeners and porous core

Dao Minh Tien¹, Nguyen Thi Thanh Hoai^{2,3,*}, Tran Quang Minh⁴

¹Air Force-Air Defense Technical Institute, Hanoi, Vietnam

²Graduate University of Science and Technology, Vietnam Academy of Science and Technology, Hanoi, Vietnam

³Faculty of International Education, University of Transport and Communications, Hanoi, Vietnam

⁴Mechanics of Advanced Materials and Structures, University of Transport Technology, Hanoi, Vietnam

Abstract: This study examines the nonlinear impulsive responses of functionally graded graphene platelet-reinforced composite (FG-GPLRC) shallow spherical shells with spiderweb stiffeners and porous cores; circular plates are included as a limiting case of the formulation. Graphene platelets are distributed in the coatings and stiffeners according to various grading patterns, while the core is modeled as a porous medium. An improved smeared stiffener technique is developed to account for the stiffness contribution of the meridian-parallel spiderweb configuration, including the stiffness merging effect near the central region. The governing equations are formulated based on Donnell shell theory and von Kármán geometric nonlinearity using the energy method. The resulting nonlinear motion equations are solved via the Runge–Kutta method under triangular (blast-type) and rectangular (step-type) impulsive loads. Parametric studies of spherical shells reveal that spiderweb stiffeners markedly increase the fundamental frequency and significantly reduce transient deflections. The graphene mass fraction, distribution pattern, porosity coefficient, and damping also play important roles in tailoring dynamic behavior. The proposed framework provides an efficient analytical tool for designing advanced FG-GPLRC structures under impulsive loading conditions.

Keywords: Nonlinear response, spherical shell, Impulsive load, functionally graded graphene platelet-reinforced composite, Energy method, Donnell shell theory.

1. Introduction

Circular plates and shallow spherical shells are classical models in nonlinear structural mechanics. Owing to curvature-induced membrane-bending coupling, spherical shells exhibit more complex buckling and vibration behaviors than flat plates, especially under large

deflection and impulsive loading where geometric nonlinearity may trigger snap-through and dynamic instability. Accurate prediction of these responses has therefore attracted continuous research interest, with various analytical and numerical approaches developed to address stability and transient dynamics. The interaction between

structural curvature, material heterogeneity, and nonlinear kinematics remains a key issue in the design of advanced plate and shell systems. Extensive investigations have been conducted on the mechanical responses of isotropic circular plates and shallow spherical shells. Kalnins and Naghdi [1] established one of the earliest exact analytical solutions for axisymmetric vibrations of shallow spherical shells, providing a rigorous theoretical foundation for subsequent dynamic studies. Bucco and Mazumdar [2] later proposed a numerical isodeflection contour method to estimate the fundamental frequencies of shallow shells, offering improved computational efficiency. The stability behavior of spherical structures has also been widely examined. Dumir et al. [3] analyzed the axisymmetric buckling of laminated thick annular spherical caps, emphasizing the role of lamination schemes and geometric parameters. Malek et al. [4] investigated the buckling characteristics of spherical cap gridshells and revisited the concept of equivalent continuum modeling. From a different theoretical perspective, Abels et al. [5] studied the stability of spherical caps under volume-preserving mean curvature flow, incorporating line tension effects into the formulation. Regarding circular plates, Caruntu and Oyervides [6] demonstrated that higher-order vibration modes are essential for accurately predicting parametric resonance in MEMS-based clamped plates. Najafipour and Shariyat [7] developed a refined first-order shear deformation theory to capture large static and dynamic deflections of hyperelastic circular plates subjected to abrupt or harmonic pressure loads. Singh and Lata [8] further extended the analysis to nonlocal thermoelastic thick plates, revealing the influence of thermal and size-dependent effects. More recently, Ardic et al. [9] incorporated fluid–structure interaction into the dynamic analysis of clamped circular plates, showing that partial fluid contact significantly alters vibration characteristics. With the increasing demand for lightweight and high-performance structures, considerable attention has

been directed toward the nonlinear behavior of functionally graded material (FGM) circular plates and spherical caps under combined thermal and mechanical loadings. Kiani [10] conducted a detailed investigation of axisymmetric static and dynamic postbuckling responses of thermally loaded FGM circular plates, providing comprehensive insight into snap-through characteristics in the thermal postbuckling regime. Saini et al. [11] examined the buckling and vibration behavior of peripherally loaded FGM circular plates subjected to nonuniform temperature distributions, highlighting the influence of thermal gradients on stability boundaries. In the context of porous FGM structures, Long and Tung [12] employed the Galerkin approach in conjunction with Donnell shell theory to analyze postbuckling responses of spherical caps and circular plates resting on nonlinear elastic foundations with edge constraints. Using first-order shear deformation theory (FSDT), Phuong et al. [13] and Minh et al. [14] explored nonlinear vibrations of sandwich FGM shallow spherical caps, considering both porous and non-porous core configurations in thermal environments. A higher-order shear deformation framework was later adopted to derive analytical solutions for thermo-mechanical buckling of porous FGM plates and caps [15], while Vasara et al. [16] utilized the differential quadrature method to evaluate free vibration characteristics of porous FGM circular and annular plates. Further investigations addressed strongly nonlinear phenomena in spherical structures. The combined effects of thermal and mechanical loads on snap-through buckling of FGM sandwich spherical caps with large curvature were analyzed in [17]. Moreover, Keibolahi et al. [18] and Javani et al. [19] studied temperature-dependent FGM spherical shells subjected to sudden thermal shocks, revealing complex dynamic snap-through and transient vibration behaviors under extreme conditions.

In recent years, graphene nanoplatelet-reinforced composite (GPLRC) structures have

attracted increasing attention due to their superior stiffness and multifunctional capabilities. Sankar et al. [20] investigated the dynamic snap-through behavior of CNT-reinforced sandwich spherical caps, demonstrating the effectiveness of nanoscale reinforcements in improving structural resistance against instability. Extending nanocomposite applications to graphene-based systems, Javani et al. [21] analyzed geometrically nonlinear free vibrations of FG-GPLRC circular plates resting on nonlinear elastic foundations, revealing the sensitivity of vibration characteristics to foundation stiffness. From a three-dimensional elasticity perspective, Liu et al. [22] derived exact solutions for free vibration and bending of graphene-reinforced spherical shells, highlighting the accuracy improvements over classical plate and shell theories. Haboussi et al. [23] employed finite element modeling combined with shear deformation theory to examine nonlinear dynamic buckling of FG-GPLRC spherical shells with porous cores, emphasizing the interaction between material gradation and porosity. Dai et al. [24] addressed thermally induced dynamic instabilities in composite circular plates and showed that softening elastic foundations can significantly reduce structural stability margins. Thermo-mechanical effects in GPLRC systems were further examined by Phuong et al. [25], who analyzed axisymmetric stability of FG-GPLRC spherical shells and circular plates under combined loading conditions. Using higher-order shear deformation theory, Ly et al. [26] explored postbuckling responses of sandwich FG-GPLRC spherical shells with porous cores, providing detailed insights into nonlinear equilibrium paths. More recently, Nam et al. [27-29] proposed an analytical formulation for nonlinear postbuckling of GPLRC circular plates and spherical shells stiffened by spiderweb stiffeners, demonstrating the effectiveness of stiffening mechanisms in enhancing buckling and postbuckling load-carrying capacity. Additionally, Wang and Jiang [30] introduced a bilateral iterative displacement control

strategy to efficiently capture nonlinear free vibration responses of dual-FG nanocomposite plates. Although significant advances have been made in the analysis of FGM and GPLRC structures, most studies focus on static stability and free vibration. Nonlinear transient responses of spiderweb-stiffened FG-GPLRC spherical shells and circular plates under impulsive loading remain largely unexplored. In particular, the stiffness merging effect and its interaction with graphene grading and porous cores have not been systematically addressed. Motivated by these limitations, the present study develops an analytical model to investigate the nonlinear impulsive responses of FG-GPLRC shallow spherical shells and circular plates reinforced by spiderweb stiffeners and integrated with porous cores. This integrated design enables a synergistic enhancement of structural performance by simultaneously reducing mass and increasing stiffness, thereby improving the stiffness-to-weight ratio and the resistance to impulsive loading. An improved smeared stiffener technique is proposed to account for the spatially varying stiffener distribution and stiffness merging effect. The governing equations are derived using Donnell shell theory and von Kármán geometric nonlinearity within an energy-based formulation. The nonlinear equations of motion are solved numerically to evaluate transient responses under blast and step impulsive loads. Comprehensive parametric studies are conducted to clarify the roles of stiffener configuration, graphene distribution, porosity coefficient, and damping on structural performance. The outcomes of this work provide deeper insight into the dynamic behavior of advanced nanocomposite shell systems and offer useful guidance for the design of impact-resistant lightweight structures.

2. Designs of FG-GPLRC spherical shells and circular plates, and main formulations

As shown in Fig. 1, the FG-GPLRC spherical shells are presented, with the thickness h , base

radius R_3 , radius of shells R_c , and spherical coordinate system (φ, θ, z) . The considered spherical shells are assumed to be axisymmetrically deformed and subjected to time-dependent impulsive pressure q . By applying the Donnell shell theory and nonlinear large deflection

assumptions, the main formulations of the nonlinear dynamic problems of the shells are obtained.

The main formulations of the FG-GPLRC circular plates can be established by taking $R_c \rightarrow \infty$.

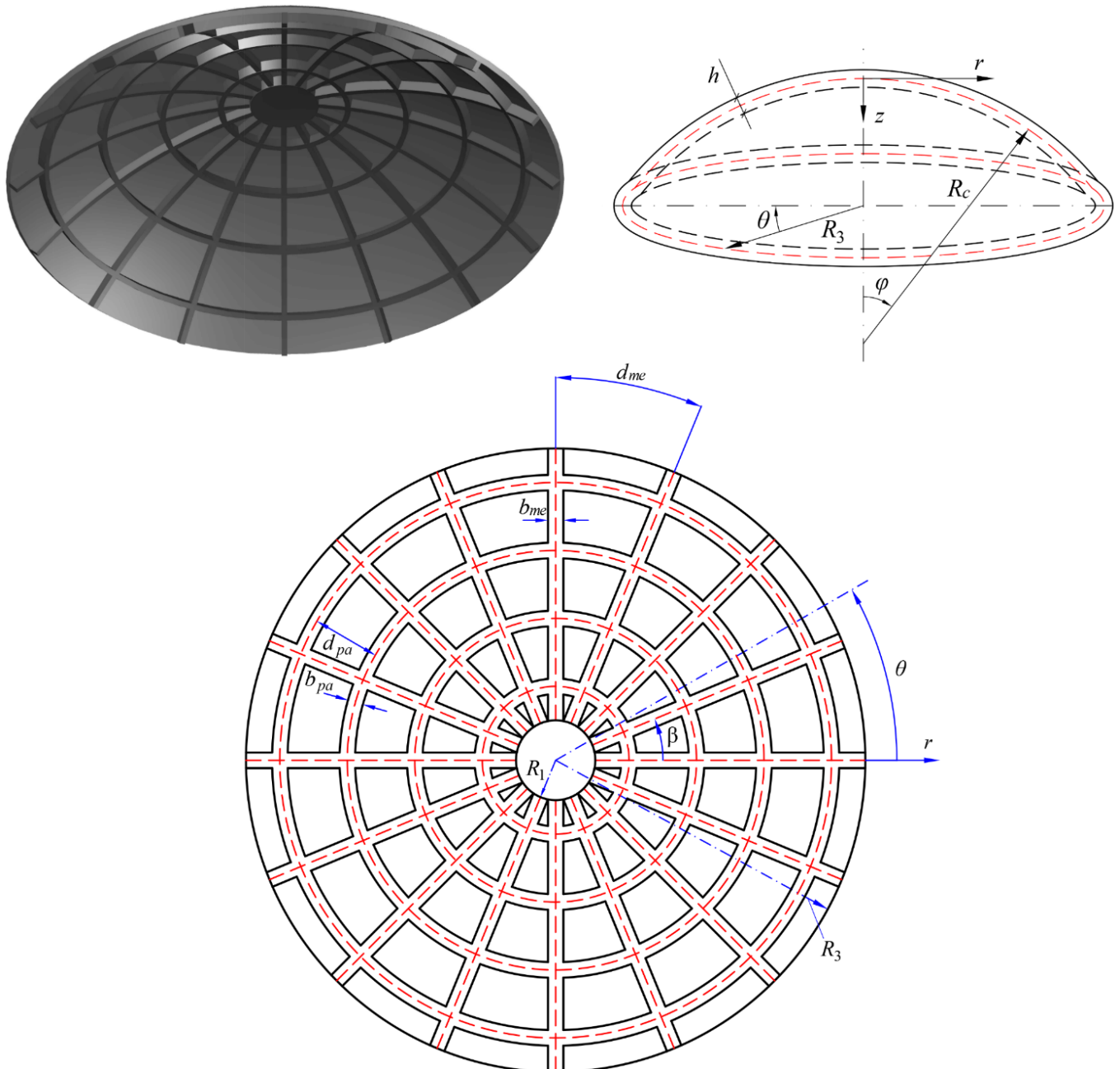


Fig. 1. Configurations and coordinate systems of FG-GPLRC spherical shells and circular plates

The shell is reinforced by meridian or parallel stiffeners or a meridian-parallel spiderweb stiffener system (Fig. 1). The angle between meridian stiffeners is given by $\beta = 2\pi/n_{me}$, where n_{me} is the

number of meridian stiffeners. The radius of the merged area is $R_1 = b_{pa}/[2\sin(\beta/2)]$, with b_{pa} representing the width of the parallel stiffeners. The distance between parallel stiffeners is

$d_{pa} = (R_3 - R_1)/n_{pa}$, and the distance between meridian stiffeners is $d_{me} = \beta r$, where n_{pa} is the number of parallel stiffeners. Here, b_{pa} and b_{me} denote the width of the parallel and meridian stiffeners, respectively, n_{pa} and n_{me} represent the number of parallel and meridian stiffeners.

In this paper, FG-GPLRC is designed for the material of the considered structures. Coatings and porous core are combined into five models as UD-PC-UD/UD, X-PC-X/X, O-PC-O/O, V-PC-ΛV, and Λ-PC-V/Λ as shown in Fig. 2.

The mass fraction distributions of graphene

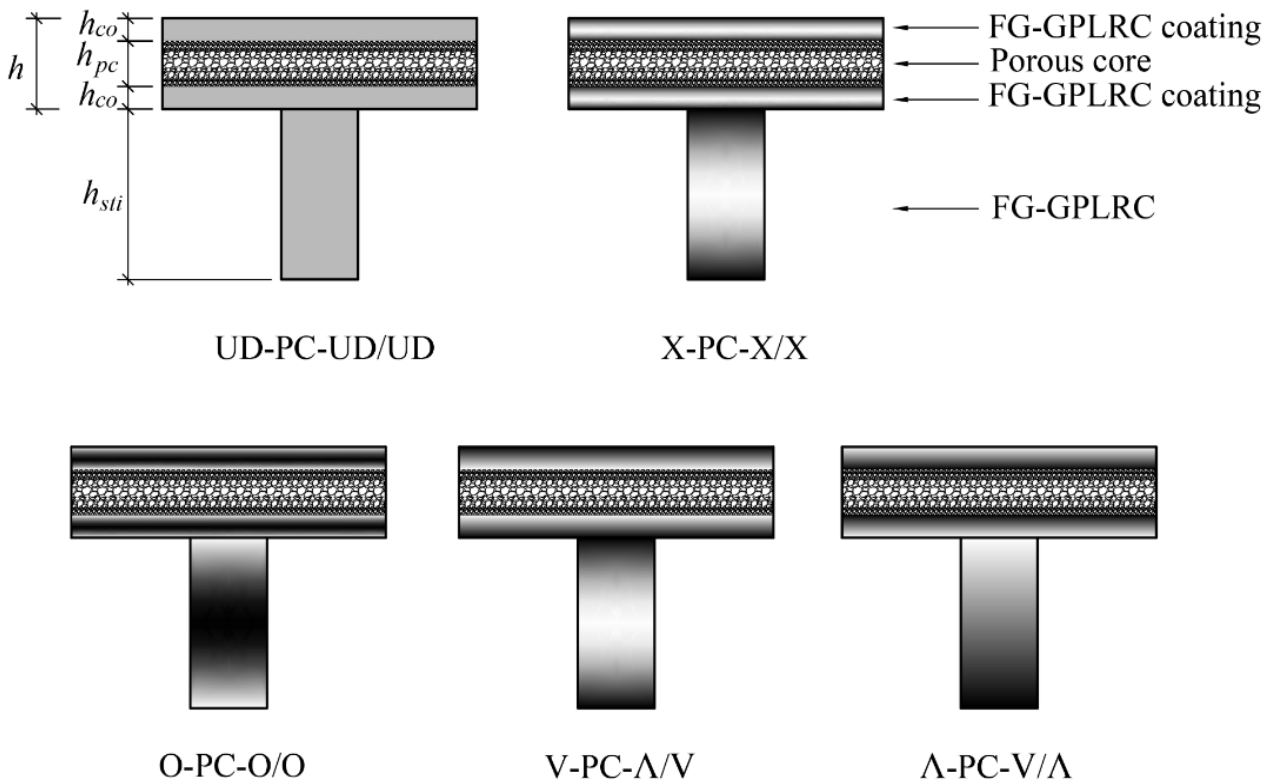


Fig. 2. Graphene and porosity distribution laws of coatings, stiffeners, and porous core

$$W_{GPL}^B = \begin{cases} W_{GPL} & \text{UD law,} \\ \left[\frac{(8z - 2h - 2h_{pc})}{(h - h_{pc})} \right] W_{GPL} & \text{X law,} \\ \left[2 - \frac{(8z - 2h - 2h_{pc})}{(h - h_{pc})} \right] W_{GPL} & \text{O law,} \\ \left[\frac{(2h - 4z)}{(h - h_{pc})} \right] W_{GPL} & \text{V law,} \\ \left[\frac{(4z - 2h_{pc})}{(h - h_{pc})} \right] W_{GPL} & \Lambda \text{ law,} \end{cases} \quad (2)$$

and for stiffeners ($h/2 \leq z \leq h/2 + h_{sti}$)

in coatings and stiffeners are considered in five laws, as follows [26]

+) The upper coating ($-h/2 \leq z \leq -h_{pc}/2$)

$$W_{GPL}^T = \begin{cases} W_{GPL} & \text{UD law,} \\ \left[\frac{(8z + 2h + 2h_{pc})}{(h - h_{pc})} \right] W_{GPL} & \text{X law,} \\ \left[2 - \frac{(8z + 2h + 2h_{pc})}{(h - h_{pc})} \right] W_{GPL} & \text{O law,} \\ \left[\frac{(-2h_{pc} - 4z)}{(h - h_{pc})} \right] W_{GPL} & \text{V law,} \\ \left[\frac{(4z + 2h)}{(h - h_{pc})} \right] W_{GPL} & \Lambda \text{ law,} \end{cases} \quad (1)$$

+) The lower coating ($h_{pc}/2 \leq z \leq h/2$)

$$W_G^{sti} = \begin{cases} W_{GPL} & \text{UD law,} \\ \left[2 \frac{(h - 2z)}{h_{sti}} + 1 \right] W_{GPL} & \text{X law,} \\ \left[2 - 2 \frac{(2z - h)}{h_{sti}} - 1 \right] W_{GPL} & \text{O law,} \\ \left[\frac{(-2z + h)}{h_{sti}} + 2 \right] W_{GPL} & \text{V law,} \\ \left[\frac{(-h + 2z)}{h_{sti}} \right] W_{GPL} & \Lambda \text{ law,} \end{cases} \quad (3)$$

where W_{GPL} is the denote of the total mass fraction of graphene.

The graphene volume fraction in the upper and lower coatings and stiffeners can be defined, respectively, as

$$V_{\text{GPL}}^T = \frac{\rho_m W_{\text{GPL}}^T}{\rho_m W_{\text{GPL}}^T + \rho_{\text{GPL}} (1 - W_{\text{GPL}}^T)}, \quad (4)$$

$$V_{\text{GPL}}^B = \frac{\rho_m W_{\text{GPL}}^B}{\rho_m W_{\text{GPL}}^B + \rho_{\text{GPL}} (1 - W_{\text{GPL}}^B)}, \quad (5)$$

$$V_{\text{GPL}}^{\text{sti}} = \frac{\rho_m W_{\text{GPL}}^{\text{sti}}}{\rho_m W_{\text{GPL}}^{\text{sti}} + \rho_{\text{GPL}} (1 - W_{\text{GPL}}^{\text{sti}})}, \quad (6)$$

where ρ_{GPL} and ρ_m are the densities of the graphene and the matrix, respectively.

The elastic moduli of the FG-GPLRC coatings and FG-GPLRC stiffeners are determined by the extended Halpin-Tsai model below.

$$E_T = \frac{E_m}{8} \left(\frac{3 + 3m_0 n_0 V_{\text{GPL}}^T}{1 - n_0 V_{\text{GPL}}^T} + \frac{5 + 5m_1 n_1 V_{\text{GPL}}^T}{1 - n_1 V_{\text{GPL}}^T} \right), \quad (7)$$

$$E_B = \frac{E_m}{8} \left(\frac{3 + 3m_0 n_0 V_{\text{GPL}}^B}{1 - n_0 V_{\text{GPL}}^B} + \frac{5 + 5m_1 n_1 V_{\text{GPL}}^B}{1 - n_1 V_{\text{GPL}}^B} \right), \quad (8)$$

$$E_{\text{sti}} = \frac{E_m}{8} \left(\frac{3 + 3m_0 n_0 V_{\text{GPL}}^{\text{sti}}}{1 - n_0 V_{\text{GPL}}^{\text{sti}}} + \frac{5 + 5m_1 n_1 V_{\text{GPL}}^{\text{sti}}}{1 - n_1 V_{\text{GPL}}^{\text{sti}}} \right), \quad (9)$$

where

$$n_0 = \frac{E_{\text{GPL}} - E_m}{E_{\text{GPL}} + m_0 E_m}, \quad n_1 = \frac{E_{\text{GPL}} - E_m}{E_{\text{GPL}} + m_1 E_m},$$

$$m_0 = \frac{2a_{\text{GPL}}}{t_{\text{GPL}}}, \quad m_1 = \frac{2b_{\text{GPL}}}{t_{\text{GPL}}},$$

with t_{GPL} , a_{GPL} , b_{GPL} are respectively the thickness, length, and width of graphene; E_m and E_{GPL} are respectively the elastic moduli of the matrix and graphene.

According to the mixture rule, the Poisson's ratios, and densities of FG-GPLRC shells are derived as follows:

+) The upper and lower coatings

$$\begin{aligned} v_T &= v_m (1 - V_{\text{GPL}}^T) + v_{\text{GPL}} V_{\text{GPL}}^T, \\ v_B &= v_m (1 - V_{\text{GPL}}^B) + v_{\text{GPL}} V_{\text{GPL}}^B, \\ \rho_T &= \rho_m (1 - V_{\text{GPL}}^T) + \rho_{\text{GPL}} V_{\text{GPL}}^T, \\ \rho_B &= \rho_m (1 - V_{\text{GPL}}^B) + \rho_{\text{GPL}} V_{\text{GPL}}^B \end{aligned} \quad (10)$$

and for stiffeners

$$\begin{aligned} v_{\text{sti}} &= v_m (1 - V_{\text{GPL}}^{\text{sti}}) + v_{\text{GPL}} V_{\text{GPL}}^{\text{sti}}, \\ \rho_{\text{sti}} &= \rho_m (1 - V_{\text{GPL}}^{\text{sti}}) + \rho_{\text{GPL}} V_{\text{GPL}}^{\text{sti}} \end{aligned} \quad (11)$$

The core layers are made of the same material as the matrix in this paper. Effectiveness parameters of the porous cores, including elastic modulus, coefficient of thermal expansion, Poisson's ratio, and density, can be predicted as [26]

$$\begin{cases} E_C = [1 - e_1 \cos(\pi z/h_{\text{pc}})] E_m, & -\frac{h_{\text{pc}}}{2} < z < \frac{h_{\text{pc}}}{2}, \\ v_C = v_m, \\ \rho_C = [1 - \cos(\pi z/h_{\text{pc}}) + \sqrt{1 - e_1} \cos(\pi z/h_{\text{pc}})] \rho_m \end{cases} \quad (12)$$

where the porosity coefficient is denoted by e_1 ($0 \leq e_1 < 1$).

According to Hooke's law, the stress-strain relations can be established, presented as

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} \quad (13)$$

where the reduced stiffnesses can be obtained for the shell/plate, as

$$\begin{aligned} Q_{11}^T &= Q_{22}^T = \frac{E_T}{1 - v_T^2}, & Q_{12}^T &= \frac{E_T v_T}{1 - v_T^2}, \\ Q_{11}^C &= Q_{22}^C = \frac{E_C}{1 - v_C^2}, & Q_{12}^C &= \frac{E_C v_C}{1 - v_C^2}, \\ Q_{11}^B &= Q_{22}^B = \frac{E_B}{1 - v_B^2}, & Q_{12}^B &= \frac{E_B v_B}{1 - v_B^2}, \end{aligned} \quad (14)$$

and for stiffeners

$$\begin{aligned} Q_{11}^{\text{sti}} &= Q_{22}^{\text{sti}} = E_{\text{sti}}, \\ Q_{12}^{\text{sti}} &= 0, \end{aligned} \quad (15)$$

According to the Donnell shell theory and axisymmetric assumption, the meridian and parallel displacements, and deflection of shells are derived as [12]

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = \begin{Bmatrix} u(r) \\ 0 \\ w(r) + w_0(r) \end{Bmatrix}, \quad (16)$$

where $w_0(r)$ is the imperfect deflection of structures, and the displacements $\bar{u} = \bar{u}(r, z)$, and

$$\bar{v} = \bar{v}(r, z), \quad \bar{w} = \bar{w}(r, z).$$

The axisymmetric strains of the shells at a distance of z from the mid-plane can be derived as

$$\{\varepsilon_r, \varepsilon_\theta\} = \left\{ \varepsilon_{r0} - z w_{,rr}, \varepsilon_{\theta 0} - \frac{z}{r} w_{,r} \right\}, \quad (17)$$

where $\varepsilon_{r0}, \varepsilon_{\theta 0}$ are the mid-plane strains of shells formulated using von Karman's nonlinear large deflection assumption, as [12]

$$\left\{ \begin{matrix} \varepsilon_{r0} \\ \varepsilon_{\theta 0} \end{matrix} \right\} = \left\{ \begin{matrix} u_{,r} - \frac{w}{R_c} + w_{,r} w_{0,r} + \frac{1}{2} w_{,r}^2 \\ -\frac{w}{R_c} + \frac{u}{r} \end{matrix} \right\}. \quad (18)$$

By applying the beam theory for FG-GPLRC stiffeners, the improved Lekhnitskii smeared stiffener technique is developed, where the stiffnesses of the anisotropic beams are distributed into the stiffnesses of the plates or shells. However, the distances between meridian stiffeners vary and merge near the center area for circular plates or spherical shells, requiring an extension of the technique to calculate the internal forces and moments, as follows.

$$\left\{ \begin{matrix} N_r \\ N_\theta \\ M_r \\ M_\theta \end{matrix} \right\} = \left\{ \begin{matrix} H_{11} & H_{12} & L_{11} & L_{12} \\ H_{12} & H_{22} & L_{12} & L_{22} \\ L_{11} & L_{12} & F_{11} & F_{12} \\ L_{12} & L_{22} & F_{12} & F_{22} \end{matrix} \right\} \left\{ \begin{matrix} \varepsilon_{r0} \\ \varepsilon_{\theta 0} \\ -w_{,rr} \\ -w_{,r}/r \end{matrix} \right\}, \quad (19)$$

where the merged area $r \in [0, R_1]$

$$\begin{aligned} (H_{11}^{in}, L_{11}^{in}, F_{11}^{in}) &= \int_{-h/2}^{-h_{pc}/2} Q_{11}^T(1, z, z^2) dz \\ &+ \int_{-h_{pc}/2}^{h_{pc}/2} Q_{11}^C(1, z, z^2) dz \\ &+ \int_{h_{pc}/2}^{h/2} Q_{11}^B(1, z, z^2) dz + \int_{h/2}^{h/2+h_{sti}} Q_{11}^{sti}(1, z, z^2) dz, \\ (H_{22}^{in}, L_{22}^{in}, F_{22}^{in}) &= \int_{-h/2}^{-h_{pc}/2} Q_{22}^T(1, z, z^2) dz \\ &+ \int_{-h_{pc}/2}^{h_{pc}/2} Q_{22}^C(1, z, z^2) dz \\ &+ \int_{h_{pc}/2}^{h/2} Q_{22}^B(1, z, z^2) dz + \int_{h/2}^{h/2+h_{sti}} Q_{22}^{sti}(1, z, z^2) dz, \end{aligned}$$

$$\begin{aligned} (H_{12}^{in}, L_{12}^{in}, F_{12}^{in}) &= \int_{-h/2}^{-h_{pc}/2} Q_{12}^T(1, z, z^2) dz \\ &+ \int_{-h_{pc}/2}^{h_{pc}/2} Q_{12}^C(1, z, z^2) dz + \int_{h_{pc}/2}^{h/2} Q_{12}^B(1, z, z^2) dz, \end{aligned}$$

and for boundary area $r \in (R_1, R_3]$

$$\begin{aligned} (H_{11}^{ou}, L_{11}^{ou}, F_{11}^{ou}) &= \int_{-h/2}^{-h_{pc}/2} Q_{11}^T(1, z, z^2) dz + \int_{-h_{pc}/2}^{h_{pc}/2} Q_{11}^C(1, z, z^2) dz \\ &+ \int_{h_{pc}/2}^{h/2} Q_{11}^B(1, z, z^2) dz + \int_{h/2}^{h/2+h_{sti}} \frac{b_{me}}{d_{me}} Q_{11}^{sti}(1, z, z^2) dz, \\ (H_{22}^{ou}, L_{22}^{ou}, F_{22}^{ou}) &= \int_{-h/2}^{-h_{pc}/2} Q_{22}^T(1, z, z^2) dz + \int_{-h_{pc}/2}^{h_{pc}/2} Q_{22}^C(1, z, z^2) dz \\ &+ \int_{h_{pc}/2}^{h/2} Q_{22}^B(1, z, z^2) dz + \int_{h/2}^{h/2+h_{sti}} \frac{b_{pa}}{d_{pa}} Q_{22}^{sti}(1, z, z^2) dz, \\ (H_{12}^{ou}, L_{12}^{ou}, F_{12}^{ou}) &= (H_{12}^{in}, L_{12}^{in}, F_{12}^{in}), \end{aligned}$$

The Lagrange function is applied as

$$\bar{\Pi} = \bar{\Pi}_T - \bar{\Pi}_{int} + \bar{\Pi}_{ext}. \quad (20)$$

where $\bar{\Pi}_{int}$, $\bar{\Pi}_{ext}$, and $\bar{\Pi}_T$ are, respectively, the strain energy, the work done by the external forces, and the kinetic energy, as

$$\bar{\Pi}_{int} = \pi \int_0^{R_1} \int_0^{2\pi} (\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta) r dr dz + \pi \int_0^{2\pi} \int_{R_1}^{R_3} (\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta) r dr dz, \quad (21)$$

$$\bar{\Pi}_{ext} = 2\pi \int_0^{R_3} q w r dr, \quad (22)$$

$$\bar{\Pi}_T = \pi \int_0^{R_1} \int_0^{2\pi} \rho(z) \bar{w}_{,t}^2 r dr dz + \pi \int_0^{2\pi} \int_{R_1}^{R_3} \rho(z) \bar{w}_{,t}^2 r dr dz, \quad (23)$$

3. Boundary condition, approximate solutions, and solving problem

In this paper, clamped boundary condition are applied for the considered structures, presented as

$$\text{At } r = R_3: \quad w = 0, \quad w_{,r} = 0, \quad u = 0, \quad (24)$$

and axisymmetrically deformed condition are also considered by

$$\text{At } r = 0: \quad u = 0, \quad w_{,r} = 0, \quad w = \text{finite}. \quad (25)$$

The conditions (24) and (25) can be satisfied if the solution forms of the displacements are presented in the polynomial functions, as [22]

$$u = U \frac{r(R_3 - r)}{R_3^2}, w = W \frac{(R_3^2 - r^2)^2}{R_3^4}, \quad (26)$$

where U and W are respectively the amplitude components.

The geometrically imperfect deflection w_0 is chosen as the same form of deflection, as

$$w_0 = \lambda h \frac{(R_3^2 - r^2)^2}{R_3^4}, \quad (27)$$

where λ is the size of the imperfection and is assumed to be small.

The Euler-Lagrange's equations and Rayleigh dissipation function are applied to obtain the motion equations, as

$$\frac{d}{dt} \left(\frac{\partial \bar{\Pi}}{\partial \dot{W}} \right) + \frac{\partial \Phi}{\partial \dot{W}} - \frac{\partial \bar{\Pi}}{\partial W} = 0, \quad \frac{d}{dt} \left(\frac{\partial \bar{\Pi}}{\partial \dot{U}} \right) - \frac{\partial \bar{\Pi}}{\partial U} = 0, \quad (28)$$

where the damping potential energy

$$\Phi = \pi \int_0^{R_3} c w_{,t}^2 r dr.$$

Substituting Equations (26) and (27) into the total energy (20), and then substituting the resultant into Equation (28), the equations of motion are obtained as

$$g_{11}U + g_{12}W + g_{13}W(W + 2\lambda h) = 0, \quad (29)$$

$$\begin{aligned} &g_{12}U + g_{22}U(W + \lambda h) \\ &+ g_{23}W(W + 4\lambda h/3) \\ &+ g_{24}W(W + \lambda h)(W + 2\lambda h) \\ &+ g_{28}W + g_{31}q \end{aligned} \quad (30)$$

$$-g_{32}\ddot{W} - g_{33}c\dot{W} = 0,$$

where g_{ij} are shown in the Appendix

From Equation (29), the amplitude U can be obtained. Then, substituting it into Eq. (30), leads to

$$\begin{aligned} &\left(g_{28} - \frac{g_{12}^2}{g_{11}} \right) W - \frac{g_{12}g_{22}}{g_{11}} W(W + \lambda h) \\ &- \frac{g_{12}g_{13}}{g_{11}} W(W + 2\lambda h) + g_{23}W \left(W + \frac{4}{3}\lambda h \right) \\ &+ \left(g_{24} - \frac{g_{13}g_{22}}{g_{11}} \right) W(W + 2\lambda h)(W + \lambda h) \end{aligned}$$

$$+ g_{31}q - g_{32}\ddot{W} - g_{33}c\dot{W} = 0. \quad (31)$$

From Eq. (31), the natural frequency of stiffened shells can be obtained by neglecting the nonlinear terms, initial imperfection, and excitation, which leads to

$$\omega = \sqrt{-\frac{g_{28} - \frac{g_{12}^2}{g_{11}}}{g_{32}}}. \quad (32)$$

It should be noted that the fundamental natural frequency obtained from Eq. (32) serves as a characteristic time scale of the structure, which is essential for defining the duration of impulsive loading and for validating the dynamic response presented in subsequent sections.

For impulsive loads, the triangular (blast) and rectangular (step) loads are considered according to the time-dependent functions, as

$$q(t) = \begin{cases} Q \left(1 - \frac{t}{T} \right), & 0 \leq t \leq T \\ 0, & t > T \end{cases} \quad \text{Blast load} \quad (33)$$

$$\begin{cases} Q, & 0 \leq t \leq T \\ 0, & t > T \end{cases} \quad \text{Step load}$$

with

$$T(s) = \frac{2\pi}{\omega} \text{ is the vibration period.}$$

4. Numerical results and discussions

Table 1 presents a comparison of the nondimensional fundamental frequency parameter of spherical shells with previously published results. The present prediction shows good agreement with the solutions reported by Kalnins and Naghdi [1], Bucco and Mazumdar [2], and Haboussi et al. [23]. The maximum deviation is relatively small, indicating the reliability and accuracy of the current formulation.

In this section, the geometric parameters used for the shells and stiffeners are specified as follows:

$$\begin{aligned} h &= 0.01\text{m}, \quad h_{pc} = 0.7h, \quad R_3 = 80h, \quad R_c = R_3/0.02, \\ h_{sti} &= 2h, \quad n_{pa} = n_{me} = 10, \quad b_{pa} = b_{me} = h, \quad \lambda = 0, \quad W_{GPL} \end{aligned}$$

= 0.5%, $e_1 = 0.3$.

It should be noted that these parameters may be modified or updated as required by the numerical analyses. The material properties employed in the present study are taken from Ref. [27].

Table 2 summarizes the fundamental frequencies of FG-GPLRC spherical shells for different graphene distribution patterns and stiffener configurations. For all material grading schemes, the presence of stiffeners markedly increases the natural frequency compared with the unstiffened case, confirming the significant stiffening effect of the reinforcement system. Among the stiffener types, the spiderweb configuration consistently yields the highest

fundamental frequency, followed by meridian and parallel stiffeners. This trend indicates that the combined meridian–parallel arrangement provides more effective in-plane and bending stiffness enhancement due to improved load redistribution and membrane–bending coupling resistance. Regarding graphene distribution patterns, variations in frequency are relatively moderate but still noticeable. The Λ -PC-V/ Λ scheme produces the highest frequencies in all stiffened cases, whereas the V-PC- Λ /V scheme yields the highest frequency in the unstiffened case. In contrast, the O-PC-O/O and V-PC- Λ /V distributions show comparatively lower values in the stiffened cases, reflecting differences in stiffness distribution through the thickness.

Table 1. Comparisons of natural frequency parameters

$$\bar{\omega} = \omega / \left(\frac{h}{R_3} \sqrt{\frac{E_m}{\rho_m R_3^2}} \right) \text{ of the spherical shells } (\bar{H}_1 = R_3^2 / 2R_c).$$

	$\bar{H}_1/h = 5$
Haboussi et al. [23]	13.37
Bucco and Mazumdar [2]	13.266
Kalnins and Naghdi [1]	13.14
Present	13.80

Table 2. Natural frequencies (rad/s) of FG-GPLRC spherical shells with different graphene distributions and stiffener configurations

	Unstiffened	Parallel stiffeners	Meridian stiffeners	Spiderweb stiffeners
UD-PC-UD/UD	251.69	321.20	326.77	359.16
X-PC-X/X	251.53	322.31	327.57	360.71
O-PC-O/O	251.39	319.36	325.23	356.75
V-PC- Λ /V	252.96	314.62	321.01	349.75
Λ -PC-V/ Λ	249.95	326.87	331.66	367.37

Table 3. Natural frequencies (rad/s) of FG-GPLRC spherical shells with different porosity coefficients (Λ -PC-V/ Λ)

e_1	0	0.2	0.4	0.6	0.8
Unstiffened	249.68	249.69	250.47	252.64	257.90
Parallel stiffeners	323.97	325.71	328.27	332.23	339.18
Meridian stiffeners	327.90	330.18	333.44	338.37	346.83
Spiderweb stiffeners	364.31	366.19	368.78	372.62	379.16

Table 3 presents the variation of fundamental frequencies with respect to the porosity coefficient for the Λ -PC-V/ Λ distribution scheme under different stiffener configurations. It can be observed that the natural frequency slightly increases as the porosity coefficient increases from 0 to 0.8 for all structural cases. This trend may appear counterintuitive, since porosity generally reduces material stiffness. However, in the present configuration, the porous core mainly affects the mass density rather than significantly degrading the global bending rigidity, which is largely

governed by the graphene-reinforced coatings and stiffeners. As the porosity coefficient increases, the reduction in structural mass becomes more pronounced than the corresponding stiffness degradation, leading to a net increase in the frequency parameter. Furthermore, the stiffened configurations consistently maintain higher frequencies than the unstiffened case across all porosity levels. The spiderweb stiffener system again exhibits the most significant enhancement, confirming its superior stiffness contribution even when the core becomes more porous.

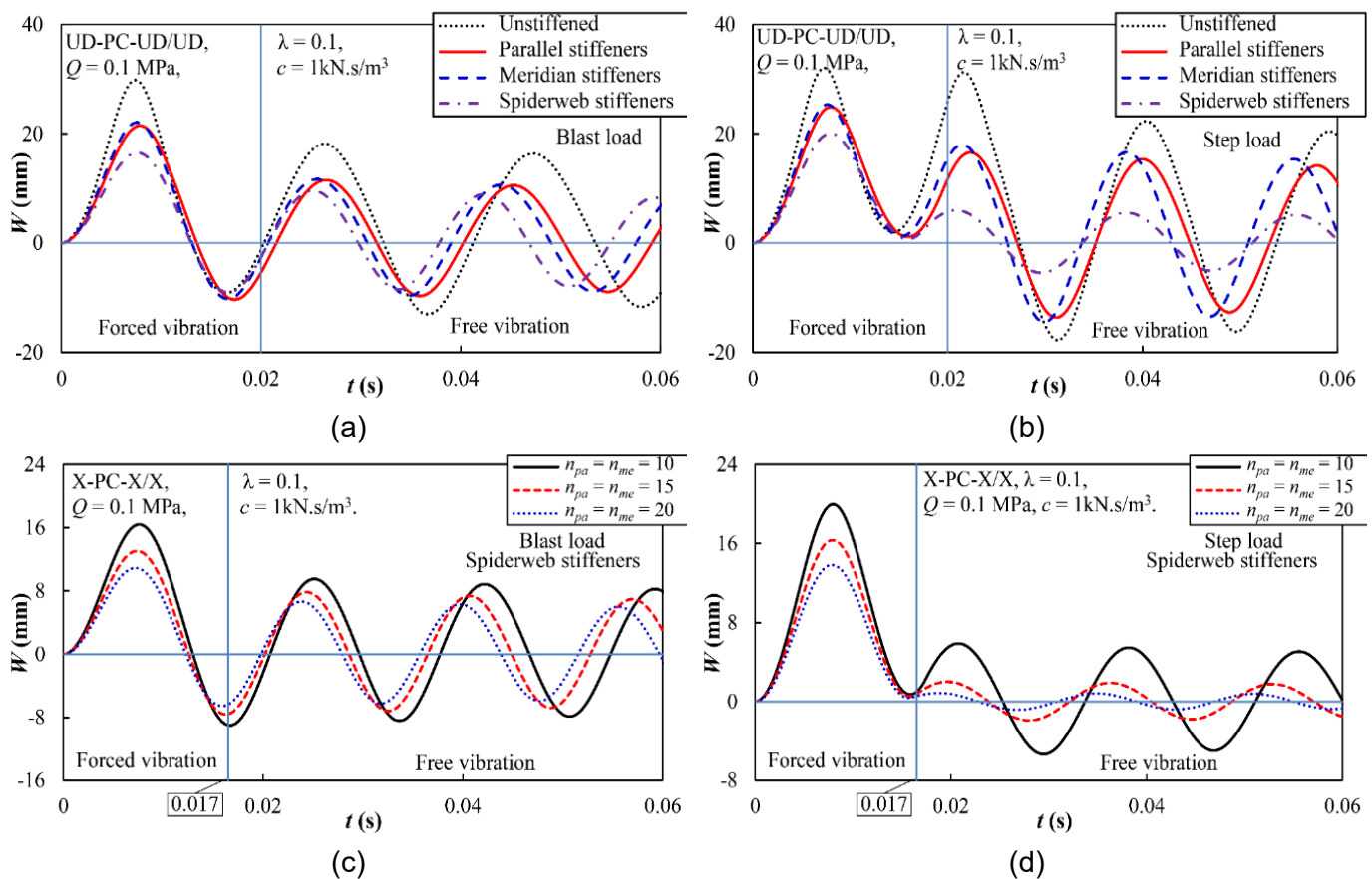


Fig. 3. Effects of stiffener types and number of stiffeners on the impulsive dynamic responses

Fig. 3 illustrates the nonlinear impulsive responses of the FG-GPLRC spherical shells, where Figs. 3(a)-(b) present the influence of stiffener configuration and Figs. 3(c)-(d) evaluate the effect of stiffener number. It can be observed that the introduction of stiffeners significantly suppresses the peak transient deflection compared with the unstiffened case, which is consistent with the frequency enhancement reported in Table 2.

Among the considered configurations, the spiderweb stiffener system produces the smallest maximum amplitude and the highest oscillation frequency, followed by meridian and parallel stiffeners. This tendency aligns with the frequency hierarchy shown in Table 2, where the spiderweb configuration consistently yields the largest fundamental frequency, confirming that increased global stiffness directly translates into improved

resistance against impulsive deformation. The superior performance of meridian stiffeners compared with parallel ones can be attributed to their stronger contribution to radial stiffness under axisymmetric loading, while the combined meridian-parallel system enhances membrane-bending coupling and load redistribution efficiency. Furthermore, as shown in Figs. 3(c)-(d), increasing the number of stiffeners leads to a progressive reduction in vibration amplitude and a shortening of the oscillation period, indicating a substantial rise

in overall structural rigidity. This behavior is physically consistent with the stiffness-dominated frequency increase observed in Tables 2 and 3.

However, the rate of amplitude reduction gradually decreases at higher stiffener densities, suggesting a diminishing return effect once the structure approaches a stiffness-saturated state. Overall, the results confirm that both stiffener configuration and density play a decisive role in governing the transient dynamic performance of the structure under impulsive loading.

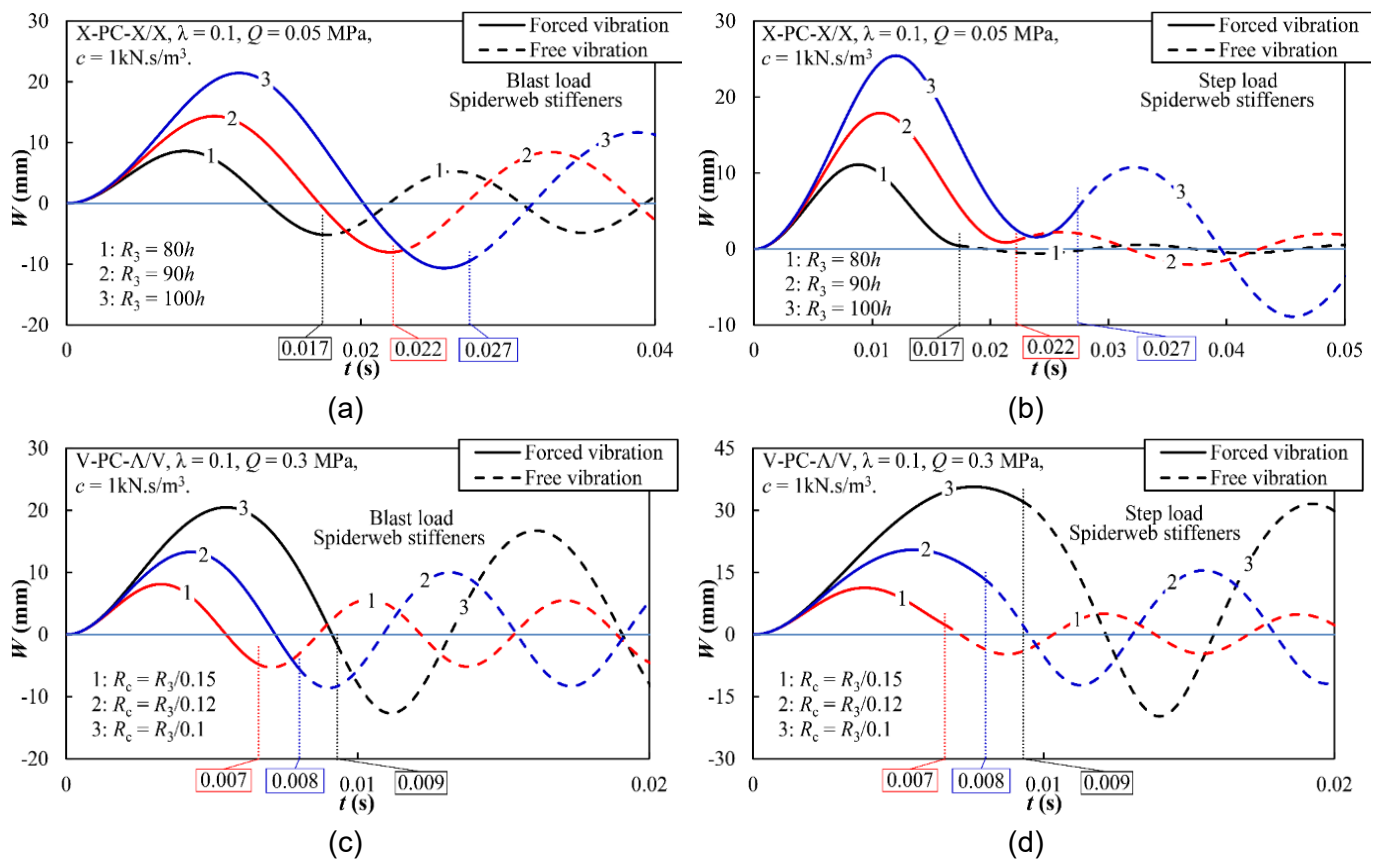


Fig. 4. Effects of geometrical parameters on the impulsive dynamic responses

Fig. 4 presents the effects of geometric parameters on the nonlinear impulsive responses of the FG-GPLRC spherical shells, where Figs. 4(a)-(b) evaluate the influence of the base radius with constant thickness, and Figs. 4(c)-(d) examine the effect of the shell radius while keeping the base radius fixed. As shown in Figs. 4(a)-(b), increasing the base radius results in a noticeable rise in the peak transient deflection and a longer oscillation period. Since the thickness remains unchanged, a larger base radius increases the structural

slenderness ratio, thereby reducing the overall bending rigidity and membrane resistance. Consequently, the shell becomes more flexible and exhibits larger dynamic amplitudes under impulsive loading.

In contrast, Figs. 4(c)-(d) indicate that increasing the shell radius (i.e., reducing curvature) while maintaining a constant base radius also leads to larger deflections and lower oscillation frequencies. This trend reflects the weakening of curvature-induced membrane-bending coupling

as the structure approaches a flatter configuration, which diminishes the inherent stiffness advantage of spherical shells. Compared with the stiffener effects discussed in Fig. 3 and the frequency variations reported in Tables 2 and 3, the geometric parameters govern the global stiffness level of the structure, thereby fundamentally controlling the magnitude and frequency of transient responses.

Fig. 5 illustrates the effects of material-related parameters on the nonlinear impulsive

responses of the FG-GPLRC spherical shells, where Figs. 5(a)-(b) evaluate the influence of graphene mass fraction, Figs. 5(c)-(d) examine the effect of the porosity coefficient, and Figs. 5(e)-(f) assess the role of the damping coefficient. As shown in Figs. 5(a)–(b), increasing the graphene mass fraction leads to a clear reduction in peak transient deflection and a slight shortening of the oscillation period. This behavior is attributed to the stiffness enhancement provided by GPLs.

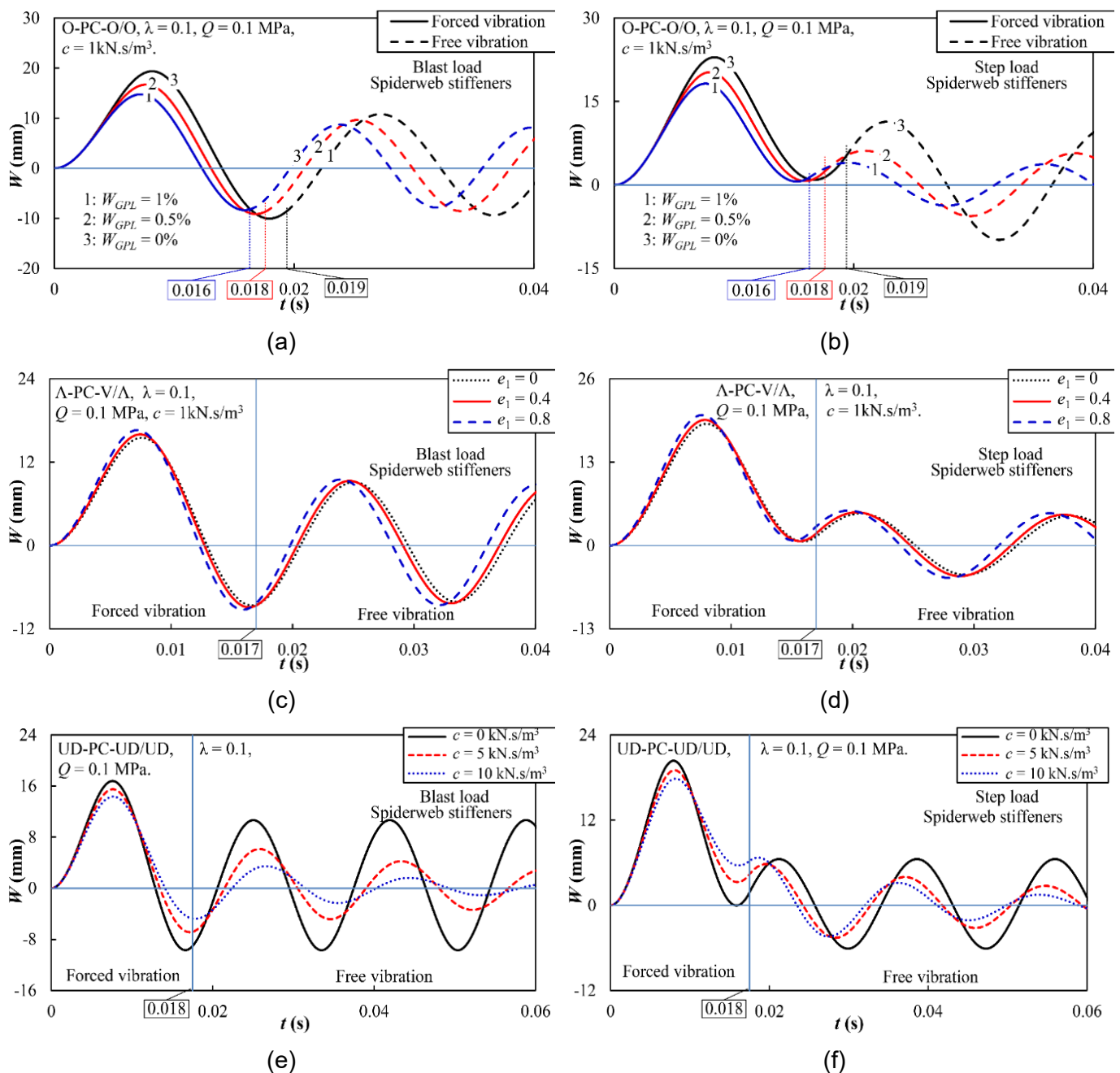


Fig. 5. Effects of graphene mass fraction, porosity coefficient, and damping coefficient on the impulsive dynamic responses

A higher graphene content increases the effective elastic modulus of the coatings and stiffeners, thereby strengthening the membrane and bending resistance under impulsive loading. In contrast, Figs. 5(c)-(d) indicate that increasing the porosity coefficient slightly reduces the vibration amplitude while marginally increasing the oscillation frequency, which is consistent with the trend observed in Table 3. Although porosity reduces core stiffness, the accompanying reduction in mass dominates the dynamic response, resulting in a higher frequency and a moderately suppressed amplitude. Finally, Figs. 5(e)-(f) demonstrate that the damping coefficient significantly affects the decay rate of oscillations rather than the initial peak response. Higher damping accelerates energy dissipation and rapidly attenuates post-impulse vibrations, while the first peak amplitude remains primarily governed by structural stiffness.

5. Conclusion

This study developed an analytical framework for investigating the nonlinear impulsive responses of FG-GPLRC shallow spherical shells reinforced by spiderweb stiffeners and integrated with porous cores; circular plates are included as a limiting case of the formulation. An improved smeared stiffener technique was formulated to incorporate the spatially varying stiffness distribution and the stiffness merging effect near the central region. The governing equations were derived based on Donnell shell theory and von Kármán geometric nonlinearity using an energy-based approach and solved numerically under blast-type and step-type impulsive loads.

The obtained results demonstrated that spiderweb stiffeners significantly enhance the fundamental frequency and effectively suppress peak transient deflections, with the spiderweb configuration providing the highest stiffness improvement.

Increasing the number of stiffeners further reduces vibration amplitude, although with

diminishing returns at high densities. Geometric parameters strongly govern global dynamic behavior, where increased slenderness or reduced curvature leads to amplified impulsive responses. Material parameters also play critical roles: higher graphene mass fractions enhance stiffness and reduce deflection; increased porosity may slightly increase frequency due to mass reduction dominance; and damping primarily accelerates vibration decay without substantially affecting the initial peak response.

Overall, the proposed model provides an efficient analytical tool for evaluating and optimizing advanced FG-GPLRC shell structures subjected to impulsive loading, offering valuable guidance for the design of lightweight and impact-resistant nanocomposite systems.

Declaration of Generative AI

During the preparation of this work, the authors did not use Generative AI and AI-assisted technology tools in order to generate the content of the manuscript.

References

- [1]. A. Kalnins, P. M. Naghdi. (1960). Axisymmetric Vibrations of Shallow Elastic Spherical Shells. *The Journal of the Acoustical Society of America*, 32(3), 342-347. <https://doi.org/10.1121/1.1908055>
- [2]. D. Bucco, J. Mazumdar. (1983). Estimation of the fundamental frequencies of shallow shells by a finite element-isodeformation contour method. *Computers & Structures*, 17(3), 441-447. [https://doi.org/10.1016/0045-7949\(83\)90135-9](https://doi.org/10.1016/0045-7949(83)90135-9)
- [3]. P.C. Dumir, G.P. Dube, A. Mallick. (2005). Axisymmetric buckling of laminated thick annular spherical cap. *Communications in Nonlinear Science and Numerical Simulation*, 10(2), 191-204. [https://doi.org/10.1016/S1007-5704\(03\)00124-2](https://doi.org/10.1016/S1007-5704(03)00124-2)
- [4]. S. Malek, T. Wierzbicki, J. Ochsendorf. (2014). Buckling of spherical cap gridshells: A numerical and analytical study revisiting the

- concept of the equivalent continuum. *Engineering Structures*, 75, 288-298. <https://doi.org/10.1016/j.engstruct.2014.05.049>
- [5]. H. Abels, H. Garcke, L. Müller. (2015). Stability of spherical caps under the volume-preserving mean curvature flow with line tension. *Nonlinear Analysis: Theory, Methods & Applications*, 117, 8-37. <https://doi.org/10.1016/j.na.2014.11.020>
- [6]. D.I. Caruntu, R. Oyervides. (2023). Amplitude–frequency response of parametric resonance of electrostatically actuated MEMS clamped circular plate. *International Journal of Non-Linear Mechanics*, 149, 104310. <https://doi.org/10.1016/j.ijnonlinmec.2022.104310>
- [7]. M. Najafipour, M. Shariyat. (2023). A refined first-order shear deformation theory for large dynamic and static deflection investigations of abruptly/harmonically pressurized/blasted incompressible circular hyperelastic plates. *Ocean Engineering*, 289(Part 1), 116185. <https://doi.org/10.1016/j.oceaneng.2023.116185>
- [8]. S. Singh, P. Lata. (2023). Effect of two temperature and nonlocality in an isotropic thermoelastic thick circular plate without energy dissipation. *Partial Differential Equations in Applied Mathematics*, 7, 100512. <https://doi.org/10.1016/j.padiff.2023.100512>
- [9]. I.T. Ardic, M.E. Yildizdag, A. Ergin. (2024). Hybrid isogeometric-based analysis and experimental investigation on the dynamic response characteristics of a clamped circular plate partially in contact with fluid. *Thin-Walled Structures*, 203, 112171. <https://doi.org/10.1016/j.tws.2024.112171>
- [10]. Y. Kiani. (2017). Axisymmetric static and dynamics snap-through phenomena in a thermally postbuckled temperature-dependent FGM circular plate. *International Journal of Non-Linear Mechanics*, 89, 1-13. <https://doi.org/10.1016/j.ijnonlinmec.2016.11.003>
- [11]. R. Saini, S. Saini, R. Lal, I.V. Singh. (2019). Buckling and vibrations of FGM circular plates in thermal environment. *Procedia Structural Integrity*, 14, 362-374. <https://doi.org/10.1016/j.prostr.2019.05.045>
- [12]. V.T. Long, H.V. Tung. (2023). Postbuckling responses of porous FGM spherical caps and circular plates including edge constraints and nonlinear three-parameter elastic foundations. *Mechanics Based Design of Structures and Machines*, 51(8), 4214-4236. <https://doi.org/10.1080/15397734.2021.1956327>
- [13]. N.T. Phuong, V.H. Nam, D.T. Dong. (2020). Nonlinear vibration of functionally graded sandwich shallow spherical caps resting on elastic foundations by using first-order shear deformation theory in thermal environment. *Journal of Sandwich Structures & Materials*, 22(4), 1157-1183. <https://doi.org/10.1177/1099636218782645>
- [14]. T.Q. Minh, D.T. Dong, V.M. Duc, N.V. Tien, N.T. Phuong, V.H. Nam. (2022). Nonlinear Axisymmetric Vibration of Sandwich FGM Shallow Spherical Caps with Lightweight Porous Core. *CIGOS 2021, Emerging Technologies and Applications for Green Infrastructure*, 23, 381-389. https://doi.org/10.1007/978-981-16-7160-9_38
- [15]. L.N. Ly, D.T.N. Thu, D.T. Dong, V.M. Duc, B.T. Tu, N.T. Phuong, V.H. Nam. (2023). A Novel Analytical Approach for Nonlinear Thermo-Mechanical Buckling of Higher-Order Shear Deformable Porous Circular Plates and Spherical Caps with FGM Face Sheets. *International Journal of Applied Mechanics*, 15(05), 2350035. <https://doi.org/10.1142/S1758825123500357>
- [16]. D. Vasara, S. Khare, H.K. Sharma, R. Kumar. (2022). Free vibration analysis of functionally graded porous circular and annular plates using differential quadrature method. *Forces in*

- Mechanics*, 9, 100126.
<https://doi.org/10.1016/j.finmec.2022.100126>
- [17]. W. Zhao, D. Guo, X. Gong, C. Li. (2023). Nonlinear axisymmetric buckling analysis of the FGM sandwich shallow spherical shells under thermomechanical loads. *European Journal of Mechanics - A/Solids*, 97, 104841.
<https://doi.org/10.1016/j.euromechsol.2022.104841>
- [18]. A. Keibolahi, Y. Kiani, M.R. Eslami. (2023). Nonlinear dynamic snap-through and vibrations of temperature-dependent FGM deep spherical shells under sudden thermal shock. *Thin-Walled Structures*, 185, 110561.
<https://doi.org/10.1016/j.tws.2023.110561>
- [19]. M. Javani, Y. Kiani, M.R. Eslami. (2024). Nonlinear dynamic response of a temperature-dependent FGM spherical shell under various boundary conditions and thermal shocks: Examination of dynamic snap-through. *Thin-Walled Structures*, 199, 111796.
<https://doi.org/10.1016/j.tws.2024.111796>
- [20]. A. Sankar, S. El-Borgi, T.B. Zineb, M. Ganapathi. (2016). Dynamic snap-through buckling of CNT reinforced composite sandwich spherical caps. *Composites Part B: Engineering*, 99, 472-482.
<https://doi.org/10.1016/j.compositesb.2016.06.027>
- [21]. M. Javani, Y. Kiani, M.R. Eslami. (2021). Geometrically nonlinear free vibration of FG-GPLRC circular plate on the nonlinear elastic foundation. *Composite Structures*, 261, 113515.
<https://doi.org/10.1016/j.compstruct.2020.113515>
- [22]. D. Liu, Y. Zhou, J. Zhu. (2021). On the free vibration and bending analysis of functionally graded nanocomposite spherical shells reinforced with graphene nanoplatelets: Three-dimensional elasticity solutions. *Engineering Structures*, 226, 111376.
<https://doi.org/10.1016/j.engstruct.2020.111376>
- [23]. M. Haboussi, A. Sankar, M. Ganapathi. (2021). Nonlinear axisymmetric dynamic buckling of functionally graded graphene reinforced porous nanocomposite spherical caps. *Mechanics of Advanced Materials and Structures*, 28(2), 127-140.
<https://doi.org/10.1080/15376494.2018.1549296>
- [24]. Z. Dai, H. Tang, S. Wu, M. Habibi, Z. Moradi, H.E. Ali. (2023). Nonlinear consecutive dynamic instabilities of thermally shocked composite circular plates on the softening elastic foundation. *Thin-Walled Structures*, 186, 110645.
<https://doi.org/10.1016/j.tws.2023.110645>
- [25]. N.T. Phuong, D.T. Dong, B.T. Tu, V.M. Duc, L.N. Khuong, P.T. Hieu, V.H. Nam. (2024). Nonlinear thermo-mechanical axisymmetric stability of FG-GPLRC spherical shells and circular plates resting on nonlinear elastic medium. *Ships and Offshore Structures*, 19(6), 820-830.
<https://doi.org/10.1080/17445302.2023.2214489>
- [26]. L.N. Ly, B.T. Tu, D.T.N. Thu, D.T. Dong, V.M. Duc, N.T. Phuong. (2023). Nonlinear thermo-mechanical buckling and postbuckling of sandwich FG-GPLRC spherical caps and circular plates with porous core by using higher-order shear deformation theory. *Journal of Thermoplastic Composite Materials*, 36(10), 4083-4105.
<https://doi.org/10.1177/08927057221147827>
- [27]. V.H. Nam, T.Q. Minh, P.T. Hieu, V.T. Hung, B.T. Tu, N.T.T. Hoai, D.T. Dong. (2023). A new analytical approach for nonlinear thermo-mechanical postbuckling of FG-GPLRC circular plates and shallow spherical caps stiffened by spiderweb stiffeners. *Thin-Walled Structures*, 193, 111296.
<https://doi.org/10.1016/j.tws.2023.111296>
- [28]. V.H. Nam, N.T.T. Hoai, N.V. Tien, P.T. Hieu,

- N.T. Phuong. (2026). Nonlinear thermo-mechanical static and dynamic buckling responses of functionally graded porous graphene-reinforced spherical shells and circular plates with spider-web stiffeners. *Journal of Thermoplastic Composite Materials*, 39(2), 702-732. <https://doi.org/10.1177/08927057251355139>
- [29]. V.H. Nam, T.Q. Minh, N.V. Tien, P.T. Hieu, N.T. Phuong. (2025). Nonlinear static and dynamic buckling analysis of GPL-reinforced spherical caps and circular plates with porous core and stepped spiderweb stiffeners. *Applied Mathematical Modelling*, 145, 116123. <https://doi.org/10.1016/j.apm.2025.116123>
- [30]. J. Wang, X. Jiang. (2024). Application of bilateral iterative displacement control method to nonlinear free vibration analysis of dual-FG nanocomposite circular plates. *Communications in Nonlinear Science and Numerical Simulation*, 138, 108236. <https://doi.org/10.1016/j.cnsns.2024.108236>

Appendix

$$g_{11} = 2\pi \int_0^{R_1} \left[\frac{(-2r + R_3)^2 H_{11}^{\text{in}} - (r - R_3)^2 H_{22}^{\text{in}}}{-2(-2r + R_3)(-r + R_3)H_{12}^{\text{in}}} \right] \frac{r}{R_3^4} dr + 2\pi \int_{R_1}^{R_3} \left[\frac{(-2r + R_3)^2 H_{11}^{\text{ou}} - (r - R_3)^2 H_{22}^{\text{ou}}}{-2(-2r + R_3)(-r + R_3)H_{12}^{\text{ou}}} \right] \frac{r}{R_3^4} dr,$$

$$g_{12} = \frac{8\pi}{R_3^6} \int_0^{R_1} \left[(R_3^2 - r^2)^2 (-2r + R_3) H_{11}^{\text{in}} / (4R_c) - (R_3^2 - r^2)^2 (3r - 2R_3) H_{12}^{\text{in}} / (4R_c) \right. \\ \left. - (r + R_3)^2 (r - R_3)^3 H_{22}^{\text{in}} / (4R_c) - (-2r + R_3)(R_3^2 - 3r^2) L_{11}^{\text{in}} - (R_3^2 - 3r^2)(-r + R_3) L_{12}^{\text{in}} \right. \\ \left. - (-2r + R_3)(R_3^2 - r^2) L_{12}^{\text{in}} - (r + R_3)(r - R_3)^2 L_{22}^{\text{in}} \right] r dr \\ + \frac{8\pi}{R_3^6} \int_{R_1}^{R_3} \left[(R_3^2 - r^2)^2 (-2r + R_3) H_{11}^{\text{ou}} / (4R_c) - (R_3^2 - r^2)^2 (3r - 2R_3) H_{12}^{\text{ou}} / (4R_c) \right. \\ \left. - (-2r + R_3)(R_3^2 - 3r^2) L_{11}^{\text{ou}} - (R_3^2 - 3r^2)(-r + R_3) L_{12}^{\text{ou}} - (-2r + R_3)(R_3^2 - r^2) L_{12}^{\text{ou}} \right. \\ \left. - (r + R_3)^2 (r - R_3)^3 H_{22}^{\text{ou}} / (4R_c) - (r + R_3)(r - R_3)^2 L_{22}^{\text{ou}} \right] r dr$$

$$g_{13} = 16\pi \int_0^{R_1} \left[\frac{(-R_3 + 2r)(r^2 - R_3^2)^2 H_{11}^{\text{in}}}{+(r - R_3)^3 (r + R_3)^2 H_{12}^{\text{in}}} \right] \frac{r^3}{R_3^{10}} dr + 16\pi \int_{R_1}^{R_3} \left[\frac{(-R_3 + 2r)(r^2 - R_3^2)^2 H_{11}^{\text{ou}}}{+(r - R_3)^3 (r + R_3)^2 H_{12}^{\text{ou}}} \right] \frac{r^3}{R_3^{10}} dr,$$

$$g_{22} = 32\pi \int_0^{R_1} \left[\frac{-(R_3^2 - r^2)^2 (-2r + R_3) H_{11}^{\text{in}}}{+(r + R_3)^2 (r - R_3)^3 H_{12}^{\text{in}}} \right] \frac{r^3}{R_3^{10}} dr + 32\pi \int_{R_1}^{R_3} \left[\frac{-(R_3^2 - r^2)^2 (-2r + R_3) H_{11}^{\text{ou}}}{+(r + R_3)^2 (r - R_3)^3 H_{12}^{\text{ou}}} \right] \frac{r^3}{R_3^{10}} dr,$$

$$g_{23} = 192\pi \int_0^{R_1} \left[(R_3^2 - r^2)^4 (H_{12}^{\text{in}} + H_{11}^{\text{in}}) / (4R_c) + (r^2 - R_3^2)^3 L_{12}^{\text{in}} - (R_3^2 - 3r^2)(R_3^2 - r^2)^2 L_{11}^{\text{in}} \right] \frac{r^3}{R_3^{12}} dr \\ + 192\pi \int_{R_1}^{R_3} \left[(R_3^2 - r^2)^4 (H_{12}^{\text{ou}} + H_{11}^{\text{ou}}) / (4R_c) + (r^2 - R_3^2)^3 L_{12}^{\text{ou}} - (R_3^2 - 3r^2)(R_3^2 - r^2)^2 L_{11}^{\text{ou}} \right] \frac{r^3}{R_3^{12}} dr,$$

$$g_{24} = 256\pi \int_0^{R_1} \left[-(r^2 - R_3^2)^4 H_{11}^{\text{in}} \right] \frac{r^5}{R_3^{16}} dr + 256\pi \int_{R_1}^{R_3} \left[-(r^2 - R_3^2)^4 H_{11}^{\text{ou}} \right] \frac{r^5}{R_3^{16}} dr,$$

$$g_{28} = 32\pi \int_0^{R_1} \left[-\pi (r^2 - R_3^2)^4 (H_{22}^{\text{in}} + 2H_{12}^{\text{in}} + H_{11}^{\text{in}}) / (16R_c^2) - (r^2 - R_3^2)^2 (3r^2 - R_3^2) (L_{11}^{\text{in}} + L_{12}^{\text{in}}) / (2R_c) \right. \\ \left. - (r^2 - R_3^2)^3 (L_{22}^{\text{in}} + L_{12}^{\text{in}}) / (2R_c) - (R_3^2 - 3r^2)^2 F_{11}^{\text{in}} - 2(R_3^4 - 4R_3^2 r^2 + 3r^4) F_{12}^{\text{in}} - (r^2 - R_3^2)^2 F_{22}^{\text{in}} \right] \frac{r}{R_3^8} dr \\ + 32\pi \int_{R_1}^{R_3} \left[-\pi (r^2 - R_3^2)^4 (H_{22}^{\text{ou}} + 2H_{12}^{\text{ou}} + H_{11}^{\text{ou}}) / (16R_c^2) - (r^2 - R_3^2)^2 (3r^2 - R_3^2) (L_{11}^{\text{ou}} + L_{12}^{\text{ou}}) / (2R_c) \right. \\ \left. - (r^2 - R_3^2)^3 (L_{22}^{\text{ou}} + L_{12}^{\text{ou}}) / (2R_c) - (R_3^2 - 3r^2)^2 F_{11}^{\text{ou}} - 2(R_3^4 - 4R_3^2 r^2 + 3r^4) F_{12}^{\text{ou}} - (r^2 - R_3^2)^2 F_{22}^{\text{ou}} \right] \frac{r}{R_3^8} dr,$$

$$g_{31} = 2\pi \int_0^{R_3} \left[(R_3^2 - r^2)^2 \right] \frac{r}{R_3^4} dr, \quad g_{33} = 2\pi \int_0^{R_3} \left[(R_3^2 - r^2)^4 \right] \frac{r}{R_3^8} dr,$$

$$g_{32} = 2\pi \int_0^{R_1} \left[(R_3^2 - r^2)^4 D_1^{\text{in}} \right] \frac{r}{R_3^8} dr + 2\pi \int_{R_1}^{R_3} \left[(R_3^2 - r^2)^4 D_1^{\text{ou}} \right] \frac{r}{R_3^8} dr,$$

$$D_1^{\text{in}} = \int_{-h/2}^{-h_{\text{pc}}/2} \rho_{\text{T}} dz + \int_{-h_{\text{pc}}/2}^{h_{\text{pc}}/2} \rho_{\text{C}} dz + \int_{h_{\text{pc}}/2}^{h/2} \rho_{\text{B}} dz + \int_{h/2}^{h/2+h_{\text{sti}}} \rho_{\text{sti}} dz,$$

$$D_1^{\text{ou}} = \int_{-h/2}^{-h_{\text{pc}}/2} \rho_{\text{T}} dz + \int_{-h_{\text{pc}}/2}^{h_{\text{pc}}/2} \rho_{\text{C}} dz + \int_{h_{\text{pc}}/2}^{h/2} \rho_{\text{B}} dz + \int_{h/2}^{h/2+h_{\text{sti}}} \frac{b_{\text{me}}}{d_{\text{me}}} \rho_{\text{sti}} dz + \int_{h/2}^{h/2+h_{\text{sti}}} \frac{b_{\text{pa}}}{d_{\text{pa}}} \rho_{\text{sti}} dz,$$