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***Corresponding author:**

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Effects of FG-GRC auxetic core on the nonlinear torsional buckling of FG-GRC toroidal shell segments in thermal environment

Pham Van Hoan¹, Bui Tien Tu^{2,*}, Nguyen Trong Tam³¹Faculty of Barrack, Military Academy of Logistics, Hanoi, Vietnam²Mechanics of Advanced Materials and Structures, University of Transport Technology, Hanoi, Vietnam³University of Transport Ho Chi Minh City, Ho Chi Minh City, Vietnam

Abstract: This paper presents an analytical study on the thermo-mechanical nonlinear buckling behavior of toroidal shell segments composed entirely of functionally graded graphene-reinforced composite (FG-GRC), including both the facesheets and the auxetic core. In contrast to conventional sandwich configurations with polymer-based cores, the present model introduces an auxetic core with a honeycomb lattice structure also made of FG-GRC, ensuring material continuity and enhanced stiffness performance. The effective material properties of the FG-GRC constituents are evaluated using an extended Halpin-Tsai model, while the auxetic core is homogenized as an equivalent orthotropic medium with functionally graded characteristics. The governing equations are formulated based on the nonlinear Donnell shell theory, incorporating von Kármán geometric nonlinearity and thermal strains induced by environmental temperature variations. A Ritz energy method is employed to derive the equilibrium equations by minimizing the total potential energy, leading to a closed-form solution for the critical torsional buckling load and postbuckling response. Parametric analyses are performed to investigate the influences of temperature change, graphene distribution patterns, geometrical parameters, and auxetic core configuration on the stability behavior. The results reveal that the fully FG-GRC configuration yields slightly higher critical torsional buckling loads and postbuckling resistance than the corresponding PMMA-core configuration, while thermal effects markedly reduce the critical buckling load.

Keywords: Toroidal shell segment, Nonlinear torsional buckling, Functionally graded graphene-reinforced composite (FG-GRC), Auxetic core, Ritz method.

1. Introduction

Revolution shells such as cylindrical shells, truncated conical shells, and toroidal shell segments are widely employed in various engineering applications due to their high load-carrying efficiency and structural stability. Among these configurations, toroidal shell segments with

shallow longitudinal curvature can be regarded as a generalized form of cylindrical shells, corresponding to the case of an infinitely large longitudinal radius. These structural components are commonly utilized in pressure vessels, fuel storage systems, underwater structures, and other advanced engineering systems. In recent years,

increasing research attention has been devoted to understanding their mechanical behavior, particularly under complex loading conditions. Moreover, the incorporation of advanced composite materials has significantly enhanced the structural performance and broadened the application potential of these shell structures. Over the past decades, extensive research has been conducted on the nonlinear static and dynamic buckling behavior of cylindrical shells made of functionally graded materials (FGMs). Various analytical and numerical approaches have been employed, including the two-step perturbation technique [1], Ritz energy method [2], Galerkin method [3, 4], and exact analytical formulations [5]. Furthermore, the structural performance of FGM cylindrical shells has been enhanced through the introduction of spiral stiffeners, developed based on an improved version of Lekhnitskii's smeared stiffener technique [6, 7]. Beyond cylindrical configurations, more geometrically complex structures such as FGM truncated conical shells have also been investigated in terms of their stability characteristics [8], while the free vibration behavior of combined cylindrical-hemispherical shell systems has been reported in the literature [9]. By employing the Stein-McElman approximation [10], the governing equations of toroidal shell segments with shallow longitudinal curvature can be simplified into a quasi-cylindrical coordinate system. This simplification has enabled several studies to analyze the nonlinear buckling response of FGM toroidal shell segments using methodologies similar to those developed for cylindrical shells [11-13].

In recent years, carbon nanotubes (CNTs) have emerged as a remarkable class of nanomaterials due to their exceptional mechanical and physical properties, making them highly suitable for reinforcing polymer matrices at the macroscale [14]. By incorporating CNTs into an isotropic matrix with a spatially varying volume fraction, Shen [15, 16] introduced the concept of functionally graded carbon nanotube-reinforced

composites (FG-CNTRCs), which exhibit enhanced and tunable material characteristics through the thickness. Subsequent studies have extensively investigated the mechanical behavior of FG-CNTRC structures. In particular, the static buckling response of cylindrical shells and panels reinforced with CNTs has been analyzed using a variety of approaches, including the two-step perturbation method [15, 16], the kp-Ritz method [17], and meshless techniques [18]. In addition to static stability, dynamic responses of CNT-reinforced cylindrical panels [19] and shells [20] have also been explored. Furthermore, the nonlinear buckling behavior of toroidal shell segments reinforced by CNTs arranged in orthogonal directions has been examined, highlighting the significant role of reinforcement orientation on structural performance [21].

Graphene has recently attracted considerable attention as an advanced nanofiller for polymer-based composites due to its outstanding mechanical properties and large surface area. In contrast to traditional functionally graded materials (FGMs) and FG-CNTRCs, functionally graded graphene-reinforced composites (FG-GRCs) are typically characterized by a layer-wise variation of graphene volume fraction through the thickness of the structure, allowing for tailored mechanical performance [22, 23]. A wide range of studies have been devoted to investigating the nonlinear stability and postbuckling behavior of FG-GRC laminated structures. These investigations cover various structural forms, including cylindrical shells [22-24], conical shells [25], plates [26], and toroidal shell segments [27]. In addition, for FG-GRC laminated shells reinforced with FG-GRC stiffeners, an enhanced smeared stiffener approach has been developed to effectively capture the stiffening effects. Using this framework, several works have examined the nonlinear buckling and postbuckling responses of stiffened FG-GRC cylindrical shells, demonstrating the significant influence of graphene distribution and stiffener configuration on structural

stability [28-30].

Auxetic materials, typically engineered in the form of lattice structures, have attracted significant interest due to their unconventional mechanical behavior, particularly their negative Poisson's ratio. This unique characteristic enables auxetic structures to exhibit enhanced energy absorption capability and improved resistance to impact loading. In addition to their superior performance under dynamic conditions, auxetic materials also offer notable advantages in static applications, especially in terms of lightweight design and structural efficiency. These features have motivated the integration of auxetic cores with advanced composite materials, leading to the development of novel sandwich structures with enhanced load-bearing capacity and mechanical performance. Consequently, a growing number of studies have focused on analyzing the mechanical behavior of plates and shell structures incorporating auxetic materials or auxetic-core configurations within composite systems [31-34].

Despite the considerable progress in the analysis of functionally graded composite shells, several important issues remain insufficiently addressed. Most existing studies on auxetic sandwich structures have primarily considered polymer-based auxetic cores, where the core mainly acts as a lightweight component with limited contribution to the overall stiffness. The potential of employing functionally graded graphene-reinforced composite (FG-GRC) materials for both the facesheets and the auxetic core has not been fully explored, particularly in terms of achieving material continuity and enhanced load transfer across the thickness. Furthermore, although the nonlinear buckling and postbuckling behaviors of FG-GRC shell structures have been extensively investigated, the majority of available works are based on Galerkin-type formulations or other approximate methods. The application of the Ritz energy method to analyze torsional buckling problems of shell structures with auxetic cores remains relatively limited, especially when thermal

effects are explicitly incorporated. In addition, while toroidal shell segments represent a class of doubly curved shells with constant principal curvatures, their nonlinear stability behavior under combined torsional and thermal loading has not been systematically studied. The interaction between auxetic geometry, graphene distribution, and thermal environment in such structures is still not well understood.

Therefore, the present study aims to develop an analytical model based on the Ritz energy method to investigate the thermo-mechanical nonlinear torsional buckling behavior of toroidal shell segments composed entirely of FG-GRC, including both the facesheets and the auxetic core. The effects of temperature variation, graphene distribution patterns, geometric parameters, and auxetic core configuration on the critical buckling load and postbuckling response are comprehensively examined.

2. Material modeling and governing equations

In the present study, toroidal shell segments subjected to torsional loading are considered, consisting of two FG-GRC laminated facesheets and an auxetic core layer. The geometrical configuration of the shell, including the FG-GRC layers and the auxetic lattice core, is illustrated in Fig. 1. The auxetic core is assumed to be fabricated from the same FG-GRC material system as the facesheets to ensure material compatibility throughout the thickness. Three graphene distribution patterns are adopted, namely UD-AU(UD)-UD, V-AU(O)- Λ , and Λ -AU(X)-V, following the formulations reported in previous studies [22, 23]. For the FG-GRC facesheets, the graphene volume fraction is assumed to vary through five layers across the thickness. Specifically, the distributions are taken as [0.11/0.09/0.07/0.05/0.03] for the FG-V pattern, [0.03/0.05/0.07/0.09/0.11] for the FG- Λ pattern, and a uniform distribution [0.07/0.07/0.07/0.07/0.07] for the UD case. Similarly, the auxetic core is modeled as a multilayer FG-GRC structure with ten layers, in

which the graphene volume fractions follow symmetric distributions. The corresponding patterns are $[0.11/0.09/0.07/0.05/0.03]_s$ for the FG-X distribution, $[0.03/0.05/0.07/0.09/0.11]_s$ for the FG-O distribution, and $[0.07/0.07/0.07/0.07/0.07]_s$

for the uniform distribution.

The effective orthotropic elastic moduli and shear modulus of the FG-GRC layers are determined based on the extended Halpin-Tsai model, given by

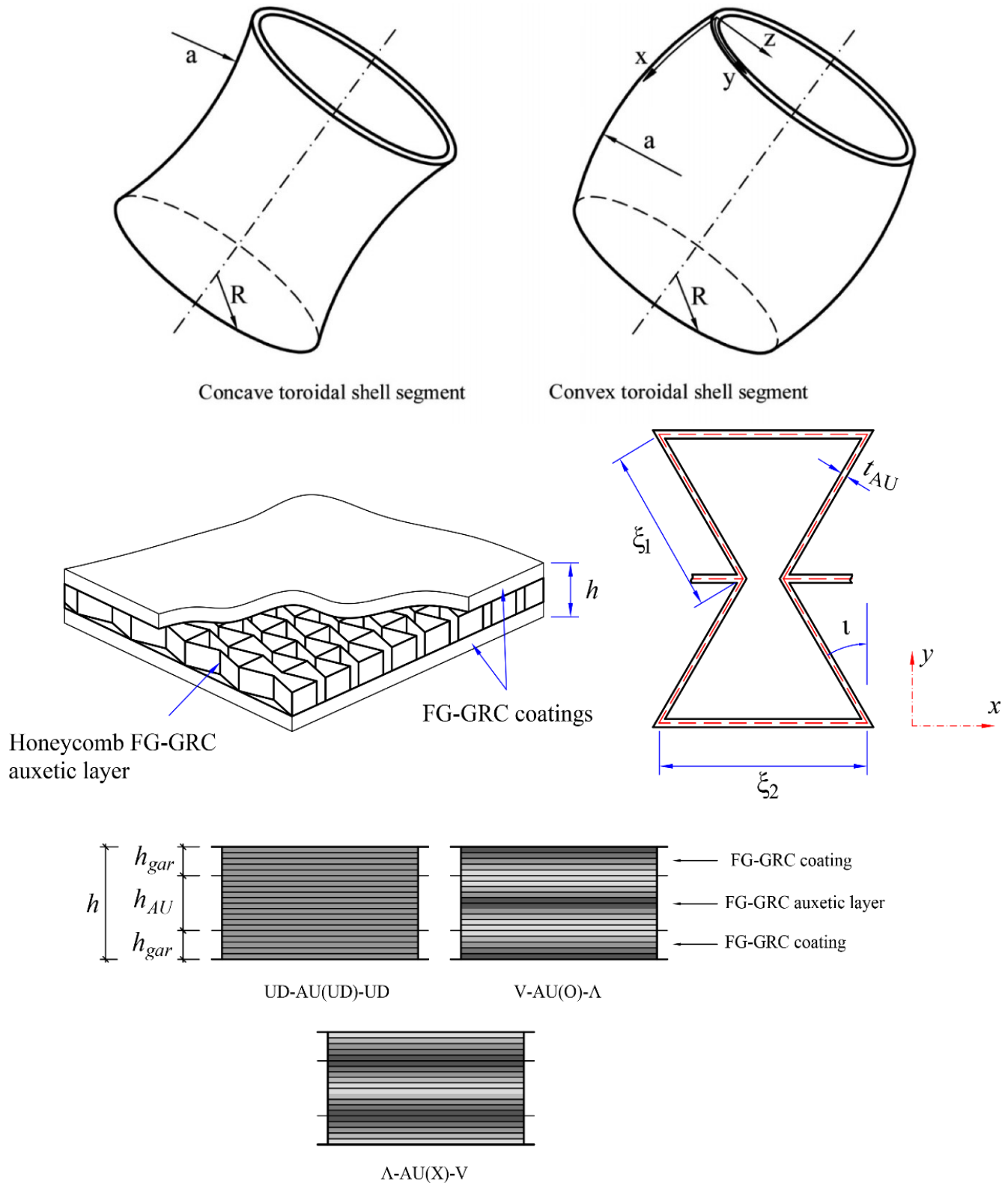


Fig. 1. Geometry and configuration of FG-GRC toroidal shell segment with auxetic core

$$\begin{aligned}
E_{11}^{gar} &= E_m \frac{(1 + 2a_{Ga}\gamma_1^{Ga}V_{Ga}/h_{Ga})\eta_1}{1 - \gamma_1^{Ga}V_{Ga}}, \\
E_{22}^{gar} &= E_m \frac{(1 + 2b_{Ga}\gamma_2^{Ga}V_{Ga}/h_{Ga})\eta_2}{1 - \gamma_2^{Ga}V_{Ga}}, \\
G_{12}^{gar} &= G_m \frac{\eta_3}{1 - \gamma_{12}^{Ga}V_{Ga}},
\end{aligned} \quad (1)$$

with

$$\begin{aligned}
\gamma_1^{Ga} &= \frac{E_{11}^{Ga}/E_m - 1}{E_{11}^{Ga}/E_m + 2a_{Ga}/h_{Ga}}, \\
\gamma_2^{Ga} &= \frac{E_{22}^{Ga}/E_m - 1}{E_{22}^{Ga}/E_m + 2b_{Ga}/h_{Ga}}, \\
\gamma_{12}^{Ga} &= \frac{G_{12}^{Ga} - G_m}{G_{12}^{Ga}},
\end{aligned} \quad (2)$$

where a_{Ga} and b_{Ga} denote the length and width of the graphene sheet, respectively, while h_{Ga} represents its effective thickness. The parameters E_{11}^{Ga} , E_{22}^{Ga} , G_{12}^{Ga} and E_m , G_m correspond to the elastic and shear moduli of the graphene sheet and the matrix material, respectively. The volume fractions of graphene and matrix are denoted by V_{Ga} and V_m , satisfying the relation $V_{Ga} + V_m = 1$. The efficiency parameters of graphene, η_j ($j = 1, 2, 3$), are dependent on the graphene volume fraction V_{Ga} .

The coefficients of thermal expansion of the FG-GRC layers in the longitudinal and circumferential directions are evaluated as follows

$$\begin{aligned}
\alpha_{11}^{gar} &= \frac{V_{Ga}E_{11}^{Ga}\alpha_{11}^{Ga} + V_mE_m\alpha_m}{V_{Ga}E_{11}^{Ga} + V_mE_m}, \\
\alpha_{22}^{gar} &= (1 + v_{12}^{Ga})V_{Ga}\alpha_{22}^{Ga} \\
&\quad + (1 + v_m)V_m\alpha_m - v_{12}^{gar}\alpha_{11}^{gar},
\end{aligned} \quad (3)$$

where α_{11}^{Ga} , α_{22}^{Ga} , and α_m denote the thermal expansion coefficients of the graphene sheet and matrix, respectively, while v_{12}^{Ga} and v_m represent their corresponding Poisson's ratios. The material properties of the matrix are assumed as $v_m = 0.34$,

$$\alpha_m = 45(1 + 0.0005\Delta T) \times 10^{-6}/K, \quad \text{and}$$

$E_m = (3.52 - 0.0034T)\text{GPa}$, where $T_0 = 300\text{K}$ denotes the reference temperature.

The effective Poisson's ratio of the FG-GRC is then evaluated based on the contributions of the graphene and matrix phases as follows

$$v_{12}^{gar} = V_{Ga}v_{12}^{Ga} + V_mv_m. \quad (4)$$

The material properties of graphene and the matrix phase used in the present study are adopted from the literature, as reported by Shen and Xiang [22, 23].

The effective properties of the FG-GRC auxetic core are determined using a two-step homogenization procedure, including thickness-wise averaging at the ligament level and structural homogenization based on the auxetic unit-cell geometry. The equivalent orthotropic elastic properties, including the Young's moduli, shear modulus, and Poisson's ratios of the FG-GRC auxetic honeycomb core, are determined as [32]

$$\begin{aligned}
E_{11}^{AU} &= E_{AU}^{gra} \frac{\kappa_3^3(\kappa_1 - \sin \iota)}{\cos^3 \iota [1 + (\tan^2 \iota + \kappa_1 \sec^2 \iota) \kappa_3^2]}, \\
E_{22}^{AU} &= E_{AU}^{gra} \frac{\kappa_3^3}{\cos \iota (\kappa_1 - \sin \iota) (\tan^2 \iota + \kappa_3^2)}, \\
G_{12}^{AU} &= E_{AU}^{gra} \frac{\kappa_3^3}{\kappa_1 (1 + 2\kappa_1) \cos \iota}, \\
v_{12}^{AU} &= -\frac{\sin \iota (1 - \kappa_3^2) (\kappa_1 - \sin \iota)}{\cos^2 \iota [1 + (\tan^2 \iota + \kappa_1 \sec^2 \iota) \kappa_3^2]}, \\
v_{21}^{AU} &= -\frac{\sin \iota (1 - \kappa_3^2)}{(\tan^2 \iota + \kappa_3^2) (\kappa_1 - \sin \iota)}, \quad \alpha_{AU} = \alpha_{AU}^{gra},
\end{aligned} \quad (5)$$

where $\kappa_1 = \xi_2/\xi_1$, $\kappa_3 = t_{AU}/\xi_1$, $E_{AU}^{gra} = \frac{1}{h_{AU}} \sum_{i=1}^{10} E_{11i}^{gar} h_{AU}^i$,

$$\alpha_{AU}^{gra} = \frac{1}{h_{AU}} \sum_{i=1}^{10} \alpha_{11i}^{gar} h_{AU}^i.$$

The shell is assumed to be shallow in the longitudinal direction, implying that the longitudinal radius a is sufficiently large. Under this condition, the Stein-McElman approximation can be employed to simplify the original doubly curved coordinate system into an equivalent quasi-

cylindrical coordinate system. In addition, the Donnell shell theory incorporating von Kármán geometric nonlinearity is adopted, as it is well suited for analyzing thin shell structures undergoing moderately large deflections.

Based on these assumptions, a combination of the nonlinear Donnell shell theory and the Stein-McElman approximation is utilized to establish the governing equations for the torsional buckling behavior of toroidal shell segments with FG-GRC facesheets and an auxetic core. Accordingly, the nonlinear strain-displacement relations can be expressed as follows [6]

$$\begin{bmatrix} \bar{\varepsilon}_x^{(0)} \\ \bar{\varepsilon}_y^{(0)} \\ \bar{\gamma}_{xy}^{(0)} \end{bmatrix} = \begin{bmatrix} u_{,x} - w/a + w_{,x}^2/2 \\ -w/R + v_{,y} + w_{,y}^2/2 \\ v_{,x} + w_{,x}w_{,y} + u_{,y} \end{bmatrix}, \quad (6)$$

where $\bar{\varepsilon}_x^{(0)}$, $\bar{\varepsilon}_y^{(0)}$, and $\bar{\gamma}_{xy}^{(0)}$ denote the mid-plane strain components, while u, v , and w represent the displacement components in the x -, y -, and z -directions, respectively. The parameters R and a correspond to the radii of the shell in the circumferential and longitudinal directions.

The strain components at an arbitrary point located at a distance z from the mid-surface are then expressed as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon}_x^{(0)} \\ \bar{\varepsilon}_y^{(0)} \\ \bar{\gamma}_{xy}^{(0)} \end{bmatrix} - z \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}, \quad (7)$$

The linear elastic constitutive relations for orthotropic materials are expressed based on Hooke's law as follows

$$\begin{aligned} \sigma_x &= \hat{Q}_{11}\varepsilon_x + \hat{Q}_{12}\varepsilon_y - \alpha_{11}(z)\Delta T, \\ \sigma_y &= \hat{Q}_{12}\varepsilon_x + \hat{Q}_{22}\varepsilon_y - \alpha_{22}(z)\Delta T, \\ \sigma_{xy} &= \hat{Q}_{66}\gamma_{xy}, \end{aligned} \quad (8)$$

where $\hat{Q}_{11} = \frac{E_{11}^{gar}}{1 - \nu_{12}^{gar}\nu_{21}^{gar}}$, $\hat{Q}_{12} = \frac{\nu_{21}^{gar}E_{11}^{gar}}{1 - \nu_{12}^{gar}\nu_{21}^{gar}}$,

$\hat{Q}_{22} = \frac{E_{22}^{gar}}{1 - \nu_{12}^{gar}\nu_{21}^{gar}}$, $\hat{Q}_{66} = G_{12}^{gar}$, for facesheets, and

$$\hat{Q}_{11} = \frac{E_{11}^{AU}}{1 - \nu_{12}^{AU}\nu_{21}^{AU}}, \quad \hat{Q}_{12} = \frac{\nu_{21}^{AU}E_{11}^{AU}}{1 - \nu_{12}^{AU}\nu_{21}^{AU}}, \quad \hat{Q}_{22} = \frac{E_{22}^{AU}}{1 - \nu_{12}^{AU}\nu_{21}^{AU}},$$

$\hat{Q}_{66} = G_{12}^{AU}$, for the auxetic core.

The stiffness contributions of the FG-GRC facesheets are obtained by integrating the Hooke relations through the thickness, while the auxetic core is first homogenized as an equivalent orthotropic layer. The overall stiffness components are then determined by assembling the contributions of all layers. The internal force and moment resultants of the FG-GRC toroidal shell segment is obtained by

$$\begin{aligned} N_x &= X_{11}\bar{\varepsilon}_x^{(0)} + X_{12}\bar{\varepsilon}_y^{(0)} - \hat{\Phi}_{1x}\Delta T, \\ N_y &= X_{12}\bar{\varepsilon}_x^{(0)} + X_{22}\bar{\varepsilon}_y^{(0)} - \hat{\Phi}_{1y}\Delta T, \quad N_{xy} = X_{66}\bar{\gamma}_{xy}^{(0)}, \\ M_x &= -Y_{11}w_{,xx} - Y_{12}w_{,yy}, \\ M_y &= -Y_{12}w_{,xx} - Y_{22}w_{,yy}, \\ M_{xy} &= -2Y_{66}w_{,xy}, \end{aligned} \quad (9)$$

where X_{ij} , Y_{ij} are the total stiffnesses, with

$$(X_{ij}, Y_{ij}) = \int_{-h/2}^{h/2} \hat{Q}_{ij}(z)(1, z^2) dz, \quad (i, j = 1, 2, 6)$$

$$\hat{\Phi}_{1x} = \int_{-h/2}^{h/2} [\hat{Q}_{11}(z)\alpha_{11}(z) + \hat{Q}_{12}(z)\alpha_{22}(z)] dz,$$

$$\hat{\Phi}_{1y} = \int_{-h/2}^{h/2} [\hat{Q}_{12}(z)\alpha_{11}(z) + \hat{Q}_{22}(z)\alpha_{22}(z)] dz.$$

3. Compatibility equation

The Airy stress function $\bar{\psi}(x, y)$ is employed to automatically satisfy the in-plane equilibrium equations, and is defined as

$$N_x = \bar{\psi}_{,yy}, \quad N_y = \bar{\psi}_{,xx}, \quad N_{xy} = -\bar{\psi}_{,xy}. \quad (10)$$

Based on Eq. (6), the compatibility equation is obtained in the following form

$$\begin{aligned} &\bar{\varepsilon}_{y,xx}^{(0)} - \bar{\gamma}_{xy,xy}^{(0)} + \bar{\varepsilon}_{x,yy}^{(0)} \\ &= w_{,xy}^2 - w_{,xx}w_{,yy} - \frac{1}{a}w_{,yy} - \frac{1}{R}w_{,xx}. \end{aligned} \quad (11)$$

By substituting Eqs. (9) and (10) into Eq. (11), the compatibility equation can be rewritten as

$$\begin{aligned} &X_{22}^*\bar{\psi}_{,xxxx} + (X_{66}^* + 2X_{12}^*)\bar{\psi}_{,xxyy} + X_{11}^*\bar{\psi}_{,yyyy} \\ &- w_{,xy}^2 + w_{,xx}w_{,yy} + \frac{1}{R}w_{,xx} + \frac{1}{a}w_{,yy} = 0. \end{aligned} \quad (12)$$

where

$$X_{11}^* = \frac{X_{22}}{X_{11}X_{22} - X_{12}^2}, X_{22}^* = \frac{X_{11}}{X_{11}X_{22} - X_{12}^2},$$

$$X_{12}^* = -\frac{X_{12}}{X_{11}X_{22} - X_{12}^2}, X_{66}^* = \frac{1}{X_{66}}.$$

4. Boundary conditions and analytical solution procedure

A sandwich toroidal shell segment with FG-GRC facesheets and an auxetic core is considered under a torsional load of intensity τ , with simply supported boundary conditions along all edges. The boundary conditions are defined as $w = 0, N_x = 0, M_x = 0, N_{xy} = -\tau h$, at $x = 0, x = L$. (13)

An approximate expression for the deflection that satisfies the prescribed boundary conditions is assumed in the following form [6, 21]

$$w = W_0 + W_1 \sin[\beta(y - \lambda x)] \sin(\alpha x) + W_2 \sin^2(\alpha x) \quad (14)$$

where $\alpha = \frac{m\pi}{L}$; $\beta = \frac{n}{R}$, with m and n denoting the axial and circumferential buckling mode numbers of the toroidal shell segment, respectively. The parameter λ represents the inclination angle of the nodal line (i.e., the deflection-vanished line) with respect to the axial direction. The coefficients W_0, W_1 , and W_2 correspond to the uniform, linear, and nonlinear components of the deflection amplitude, respectively.

The assumed deflection function in Eq. (14) follows a standard trigonometric form commonly used in analytical approaches, which approximately satisfies the boundary conditions and captures the dominant buckling modes [6, 21].

Substitution of the assumed deflection function (Eq. (14)) into the compatibility equation (Eq. (12)) yields the following expression for the stress function

$$\bar{\psi} = \bar{\psi}_1 \cos(2\alpha x) + \bar{\psi}_2 \cos[2\beta(y - \lambda x)] + \bar{\psi}_3 \cos[\beta y + (\alpha - \lambda\beta)x] + \bar{\psi}_4 \cos[\beta y - (\beta\lambda + \alpha)x] \quad (15)$$

$$+ \bar{\psi}_5 \cos[\beta y - (3\alpha + \lambda\beta)x] + \bar{\psi}_6 \cos[\beta y + (3\alpha - \lambda\beta)x] + \sigma_{oy} h \frac{x^2}{2} - \tau h x y,$$

where

$$\bar{\psi}_1 = a_{12} W_1^2 + a_{11} W_2, \quad \bar{\psi}_2 = a_{21} W_1^2,$$

$$\bar{\psi}_3 = a_{32} W_1 W_2 + a_{31} W_1, \quad \bar{\psi}_4 = a_{42} W_1 W_2 + a_{41} W_1,$$

$$\bar{\psi}_5 = a_5 W_1 W_2, \quad \bar{\psi}_6 = a_6 W_1 W_2,$$

$$a_{11} = -\frac{1}{8R\alpha^2 X_{22}^*},$$

$$a_{12} = \frac{\beta^2}{32\alpha^2 X_{22}^*},$$

$$a_{21} = \frac{\alpha^2}{32\beta^2 [\lambda^4 X_{22}^* + \lambda^2 (X_{66}^* + 2X_{12}^*) + X_{11}^*]},$$

$$a_{32} = \frac{\alpha^2 \beta^2}{2 \left[\beta^4 X_{11}^* + (X_{66}^* + 2X_{12}^*) \beta^2 (\beta\lambda - \alpha)^2 + X_{22}^* (\beta\lambda - \alpha)^4 \right]},$$

$$a_{31} = -\frac{\frac{(\beta\lambda - \alpha)^2}{R} + \frac{\beta^2}{a}}{2 \left[\beta^4 X_{11}^* + (X_{66}^* + 2X_{12}^*) \beta^2 (\beta\lambda - \alpha)^2 + X_{22}^* (\beta\lambda - \alpha)^4 \right]},$$

$$a_{42} = \frac{-\alpha^2 \beta^2}{2 \left[\beta^4 X_{11}^* + \beta^2 (\beta\lambda + \alpha)^2 (X_{66}^* + 2X_{12}^*) + (\beta\lambda + \alpha)^4 X_{22}^* \right]},$$

$$a_{41} = \frac{\frac{(\beta\lambda + \alpha)^2}{R} + \frac{\beta^2}{a}}{2 \left[\beta^4 X_{11}^* + \beta^2 (\beta\lambda + \alpha)^2 (X_{66}^* + 2X_{12}^*) + (\beta\lambda + \alpha)^4 X_{22}^* \right]},$$

$$a_5 = \frac{\alpha^2 \beta^2}{2 \left[\beta^4 X_{11}^* + \beta^2 (\beta\lambda + 3\alpha)^2 (X_{66}^* + 2X_{12}^*) + (\beta\lambda + 3\alpha)^4 X_{22}^* \right]},$$

$$a_6 = \frac{-\alpha^2 \beta^2}{2 \left[\beta^4 X_{11}^* + \beta^2 (\beta\lambda - 3\alpha)^2 (X_{66}^* + 2X_{12}^*) + (\beta\lambda - 3\alpha)^4 X_{22}^* \right]}.$$

The terms σ_{oy} and τ are introduced into the

Airy stress function as additional membrane stress components to satisfy, respectively, the circumferential closure condition and the boundary condition.

Furthermore, the closure condition of the toroidal shell segment can be formulated in terms of an average expression as [2]

$$\int_0^{2\pi R} \int_0^L v_{,y} dx dy = \int_0^{2\pi R} \int_0^L \left(\hat{\varepsilon}_y^0 + \frac{W}{R} - \frac{W_{,y}^2}{2} \right) dx dy = 0 \quad (16)$$

Based on Eq. (16), the circumferential stress σ_{0y} can be derived after some algebraic manipulations as

$$\sigma_{0y} = \frac{\left[\begin{aligned} &(-8\hat{\Phi}_{1x} X_{12}^* - 8\hat{\Phi}_{1y} X_{22}^*) \Delta T R^2 \\ &+ (-8W_0 - 4W_2) R \\ &+ n^2 W_1^2 \end{aligned} \right]}{8R^2 X_{22}^* h} \quad (17)$$

The total potential energy of the structure is formulated as

$$\hat{\Delta}_{\text{Total}} = \hat{\Delta}_{\text{in}} - \hat{\Delta}_{\text{ext}} \quad (18)$$

The elastic strain energy of the shell structure is defined as

$$\hat{\Delta}_{\text{in}} = \frac{1}{2} \int_{-h/2}^{h/2} \int_0^{2\pi R} \int_0^L \begin{bmatrix} \sigma_{xy} \gamma_{xy} + \sigma_x \varepsilon_x + \sigma_y \varepsilon_y \\ -\sigma_x \alpha_{11}(z) \Delta T \\ -\sigma_y \alpha_{22}(z) \Delta T \end{bmatrix} dx dy dz \quad (19)$$

The work of external forces induced by the applied torsional load is given by

$$\hat{\Delta}_{\text{ext}} = \int_0^{2\pi R} \int_0^L \tau h (u_{,y} + v_{,x}) dx dy \quad (20)$$

By applying the Ritz energy method, the stationary conditions of the total potential energy are obtained as

$$\frac{\partial \hat{\Delta}_{\text{Total}}}{\partial W_0} = 0, \quad \frac{\partial \hat{\Delta}_{\text{Total}}}{\partial W_1} = 0, \quad \frac{\partial \hat{\Delta}_{\text{Total}}}{\partial W_2} = 0 \quad (21)$$

By combining Eqs. (17) and (21), the governing equations can be obtained as

$$\bar{H}_{11}(2W_0 + W_2) + \bar{H}_{12} W_1^2 + \bar{H}_{14} \Delta T = 0 \quad (22)$$

$$\begin{aligned} &2\bar{H}_{12} W_0 W_1 + \bar{H}_{21} W_1^3 + \bar{H}_{22} W_1 W_2^2 \\ &+ \bar{H}_{23} W_1 W_2 + \bar{H}_{24} W_1 + \bar{H}_{25} \tau W_1 + \bar{H}_{26} W_1 \Delta T = 0 \end{aligned} \quad (23)$$

$$\begin{aligned} &\bar{H}_{11} W_0 + \bar{H}_{32} W_2 \\ &+ \bar{H}_{22} W_1^2 W_2 + \frac{\bar{H}_{23}}{2} W_1^2 + \frac{\bar{H}_{14}}{2} \Delta T = 0 \end{aligned} \quad (24)$$

where

$$\bar{H}_{11} = \frac{L\pi}{RX_{22}^*},$$

$$\bar{H}_{12} = -\frac{L\pi\beta^2}{4X_{22}^*},$$

$$\bar{H}_{14} = \frac{2L\pi(X_{12}^* \hat{\Phi}_{1x} + X_{22}^* \hat{\Phi}_{1y})}{X_{22}^*},$$

$$\bar{H}_{21} = 32R(a_{21}^2 \beta^4 \lambda^4 + a_{12}^2 \alpha^4) L\pi X_{22}^*$$

$$+ 32R\beta^4 \left[(2X_{12}^* + X_{66}^*) \lambda^2 + X_{11}^* \right] a_{21}^2 L\pi + \frac{R\beta^4 L\pi}{16X_{22}^*},$$

$$\bar{H}_{22} = RL\pi \left\{ \begin{aligned} &\left[X_{22}^* \lambda^4 + (2X_{12}^* + X_{66}^*) \lambda^2 + X_{11}^* \right] \\ &\times (a_{32}^2 + a_{42}^2 + a_5^2 + a_6^2) \beta^4 \\ &+ (6a_5^2 - 6a_6^2 - 2a_{32}^2 + 2a_{42}^2) \\ &\times \left[2X_{22}^* \lambda^3 + (2X_{12}^* + X_{66}^*) \lambda \right] \alpha \beta^3 \\ &+ (6X_{22}^* \lambda^2 + 2X_{12}^* + X_{66}^*) \\ &\times (9a_5^2 + 9a_6^2 + a_{32}^2 + a_{42}^2) \alpha^2 \beta^2 \\ &+ (108a_5^2 - 108a_6^2 - 4a_{32}^2 + 4a_{42}^2) \alpha^3 X_{22}^* \lambda \beta \\ &+ \alpha^4 X_{22}^* (81a_5^2 + 81a_6^2 + a_{32}^2 + a_{42}^2) \end{aligned} \right\},$$

$$\bar{H}_{23} = 2LR\pi \left\{ \begin{aligned} &(a_{31} a_{32} + a_{41} a_{42}) (\lambda^4 \beta^4 + 6\alpha^2 \beta^2 \lambda^2) X_{22}^* \\ &+ \alpha^4 (16a_{11} a_{12} + a_{31} a_{32} + a_{41} a_{42}) X_{22}^* \\ &+ (a_{41} a_{42} - a_{31} a_{32}) (4\alpha \lambda^3 \beta^3 + 4\alpha^3 \beta \lambda) X_{22}^* \\ &+ (a_{31} a_{32} + a_{41} a_{42}) \\ &\times \left[(2X_{12}^* + X_{66}^*) (\lambda^2 \beta^4 + \alpha^2 \beta^2) + X_{11}^* \beta^4 \right] \\ &+ 2\lambda (-a_{31} a_{32} + a_{41} a_{42}) \\ &\times (2X_{12}^* + X_{66}^*) \alpha \beta^3 - \beta^2 / (8RX_{22}^*) \end{aligned} \right\},$$

$$\bar{H}_{25} = -\pi R \beta^2 h L \lambda,$$

$$\bar{H}_{26} = -\frac{LR\pi (\hat{\Phi}_{1y} X_{22}^* + X_{12}^* \hat{\Phi}_{1x}) \beta^2}{2X_{22}^*},$$

$$\bar{H}_{32} = LR\pi \left(\frac{1}{2R^2 X_{22}^*} + 16a_{11}^2 \alpha^4 X_{22}^* - 4Y_{11} \alpha^4 \right).$$

$$\bar{H}_{24} = L\pi R \left\{ \begin{aligned} & \left(X_{22}^* a_{31}^2 + X_{22}^* a_{41}^2 - Y_{11}/2 \right) (\lambda^4 \beta^4 + \alpha^4) \\ & - Y_{22} \beta^4 / 2 + X_{11}^* a_{31}^2 \beta^4 + X_{11}^* a_{41}^2 \beta^4 \\ & + \left[(2X_{12}^* + X_{66}^*) (a_{31}^2 + a_{41}^2) - Y_{12} - 2Y_{66} \right] \\ & \times (\lambda^2 \beta^4 + \alpha^2 \beta^2) \\ & - \left[(4X_{22}^* \lambda^2 + 4X_{12}^* + 2X_{66}^*) \lambda \alpha \beta^3 \right] \\ & + 4\lambda X_{22}^* \alpha^3 \beta \\ & \times (a_{31} - a_{41}) (a_{31} + a_{41}) \\ & + (6X_{22}^* a_{31}^2 + 6X_{22}^* a_{41}^2 - 3Y_{11}) \alpha^2 \lambda^2 \beta^2 \end{aligned} \right\},$$

From Eqs. (22)–(24), the torsional load τ is derived as a function of the linear postbuckling amplitude W_1 , given by

$$\begin{aligned} \tau = & \frac{2 \left[\bar{H}_{23} (\bar{H}_{12} - \bar{H}_{23}/2) \bar{H}_{11} \right]}{\bar{H}_{11} \bar{H}_{25} (-2\bar{H}_{22} W_1^2 + \bar{H}_{11} - 2\bar{H}_{32})} W_1^2 \\ & - \frac{\bar{H}_{11} \bar{H}_{21} W_1^2 + \bar{H}_{11} \bar{H}_{24}}{\bar{H}_{11} \bar{H}_{25}} \\ & - \frac{\bar{H}_{11} \bar{H}_{22} W_1^4 (-\bar{H}_{23} + \bar{H}_{12})^2}{\bar{H}_{11} \bar{H}_{25} (-2\bar{H}_{22} W_1^2 + \bar{H}_{11} - 2\bar{H}_{32})^2} \\ & + \frac{(\bar{H}_{12} \bar{H}_{14} - \bar{H}_{26} \bar{H}_{11}) \Delta T}{\bar{H}_{11} \bar{H}_{25}} \end{aligned} \quad (25)$$

It is evident from Eq. (25) that bifurcation buckling occurs. Accordingly, the critical torsional buckling load is obtained by imposing the condition $W_1 \rightarrow 0$ in Eq. (25), which yields

$$\tau_{cr} = \frac{(\bar{H}_{12} \bar{H}_{14} - \bar{H}_{26} \bar{H}_{11}) \Delta T - \bar{H}_{11} \bar{H}_{24}}{\bar{H}_{11} \bar{H}_{25}} \quad (26)$$

According to Eq. (14), the maximum deflection of the shell consists of three amplitude components. Utilizing Eqs. (22)–(24), this expression can be rewritten in terms of the linear postbuckling amplitude as

$$W_{max} = W_0 + W_1 + W_2 = \frac{-\bar{H}_{12} W_1^2 - \bar{H}_{14} \Delta T}{2\bar{H}_{11}}$$

$$- \frac{W_1^2 (-\bar{H}_{23} + \bar{H}_{12})}{-4\bar{H}_{22} W_1^2 + 2\bar{H}_{11} - 4\bar{H}_{32}} + W_1. \quad (27)$$

The twist angle of the shell is evaluated based on an average formulation as

$$\varphi = \frac{1}{2\pi RL} \int_0^{2\pi R} \int_0^L (v_{,x} + u_{,y}) dx dy = \frac{W_1^2 n^2 \lambda}{4R^2} + \tau h X_{66}^*. \quad (28)$$

The postbuckling behavior of the sandwich toroidal shell segment with FG-GRC facesheets and an auxetic core is investigated through the torsion-deflection relationship, obtained by combining Eqs. (25) and (27). Similarly, the postbuckling torsion-twist angle relationship of the shell can be established by combining Eqs. (25) and (28).

5. Numerical results and parametric study

Table 1 confirms the validity of the present model through comparison with the results of Shen and Xiang [23]. A good agreement is observed for all lay-up configurations and graphene distribution patterns. The differences between the two sets of results are small, which verifies the correctness of the present formulation.

Table 2 presents the critical torsional buckling loads of FG-GRC toroidal and cylindrical shell segments with auxetic cores under different graphene distribution patterns. It can be observed that the shell geometry has a significant influence on the buckling behavior. Specifically, the convex shell configuration yields the highest critical torsional loads, followed by the cylindrical shell, while the concave shell exhibits the lowest values. This trend indicates that positive curvature enhances the torsional stiffness and stability of the shell, whereas negative curvature reduces its load-carrying capacity. In addition, the influence of graphene distribution is clearly evident. For all shell configurations, the UD-AU(UD)-UD pattern produces the highest critical loads, while the Λ -AU(X)-V pattern results in the lowest values. Furthermore, a comparison between FG-GRC and

PMMA auxetic cores shows that the FG-GRC core consistently provides slightly higher critical torsional buckling loads. This improvement can be attributed to the superior mechanical properties of FG-GRC, which enhance the load transfer and stiffness of the sandwich structure.

Table 1. Comparison of critical torsional buckling moments $M_{cr} = 2\pi h R^2 \tau_{cr}$ (kNm) of FG-GRC laminated cylindrical shells with the results of Shen and Xiang [23] ($R/h = 30$, $h = 2\text{mm}$, $L^2/Rh = 400$, $T = 300\text{K}$).

Direction arrangement	Shen and Xiang [23]		Present	
	UD	FG-X	UD	FG-X
$(0)_{10}$	18.98	20.58	19.45	21.13
$(0/90/0/90/0)_S$	18.99	20.48	19.47	21.15
$(0/90)_{5T}$	19.04	20.62	19.48	21.18

Table 2. Comparison of critical torsional buckling loads of FG-GRC shells with FG-GRC and PMMA auxetic cores under various graphene distribution patterns (MPa, $h = 2\text{mm}$, $h_{AU} = 0.8h$, $T = 400\text{K}$, $R = 80h$, $L = 1.5R$, $m = 1$, $\kappa_1 = 2$, $\kappa_3 = 0.1$, $\iota = 30^\circ$)

		UD-AU(UD)-UD	V-AU(O)- Λ	Λ -AU(X)-V
FG-GRC Auxetic core	Convex shell ($a = 4R$)	88.26 (8;0.62) ^a	83.99 (8;0.61)	80.30 (8;0.61)
	Cylindrical shell ($a = \infty$)	62.31 (7;0.51)	59.25 (7;0.50)	56.26 (7;0.50)
	Concave shell ($a = -4R$)	54.29 (6;0.65)	51.75 (6;0.64)	49.38 (7;0.61)
PMMA Auxetic core	Convex shell ($a = 4R$)	86.99 (8;0.60)	82.76 (8;0.60)	79.08 (8;0.60)
	Cylindrical shell ($a = \infty$)	61.77 (7;0.50)	58.75 (7;0.49)	55.75 (7;0.49)
	Concave shell ($a = -4R$)	53.68 (6;0.63)	51.14 (6;0.62)	48.84 (6;0.62)

^a The buckling modes ($n; \lambda$)

Table 3. Effect of temperature on the critical torsional buckling loads of FG-GRC toroidal shell segments with auxetic core under different graphene distribution patterns (MPa, $h = 2\text{mm}$, $h_{AU} = 0.8h$, $R = 80h$, $L = 1.5R$, $m = 1$, $\kappa_1 = 2$, $\kappa_3 = 0.1$, $\iota = 30^\circ$)

	T (K)	UD-AU(UD)-UD	V-AU(O)- Λ	Λ -AU(X)-V
Convex shell ($a = 4R$)	300	101.29 (8;0.61)	97.30 (8;0.60)	92.90 (8;0.60)
	400	88.26 (8;0.62)	83.99 (8;0.61)	80.30 (8;0.61)
	500	81.61 (7;0.65)	79.20 (8;0.62)	75.76 (8;0.62)
Concave shell ($a = -4R$)	300	62.37 (6;0.64)	60.04 (6;0.63)	56.79 (7;0.60)
	400	54.29 (6;0.65)	51.75 (6;0.64)	49.38 (7;0.61)
	500	50.17 (6;0.66)	48.74 (6;0.65)	46.59 (6;0.65)

Table 3 illustrates the influence of temperature on the critical torsional buckling loads of FG-GRC toroidal shell segments with auxetic cores. It is clearly observed that the critical buckling load decreases significantly with increasing temperature for all shell configurations and graphene distribution patterns. For instance, in the convex shell with UD-AU(UD)-UD distribution, the

critical load reduces from 101.29 MPa at $T = 300\text{K}$ to 81.61 MPa at $T = 500\text{K}$, indicating a substantial reduction in structural stability due to thermal effects. A similar trend is observed for concave shells, where the critical loads decrease from 62.37 MPa to 50.17 MPa over the same temperature range. This behavior can be attributed to the temperature-dependent degradation of material stiffness, particularly the reduction in the effective elastic moduli of the FG-GRC material, which weakens the load-carrying capacity of the shell. Moreover, the influence of graphene distribution

remains consistent across all temperatures. The UD-AU(UD)-UD pattern yields the highest critical buckling loads, while the Λ -AU(X)-V pattern produces the lowest values. This indicates that although temperature significantly affects the magnitude of the buckling load, the relative effectiveness of different graphene distribution patterns is preserved. In addition, the convex shell consistently exhibits higher critical loads than the concave shell, confirming that positive curvature enhances torsional stability even under thermal environments.

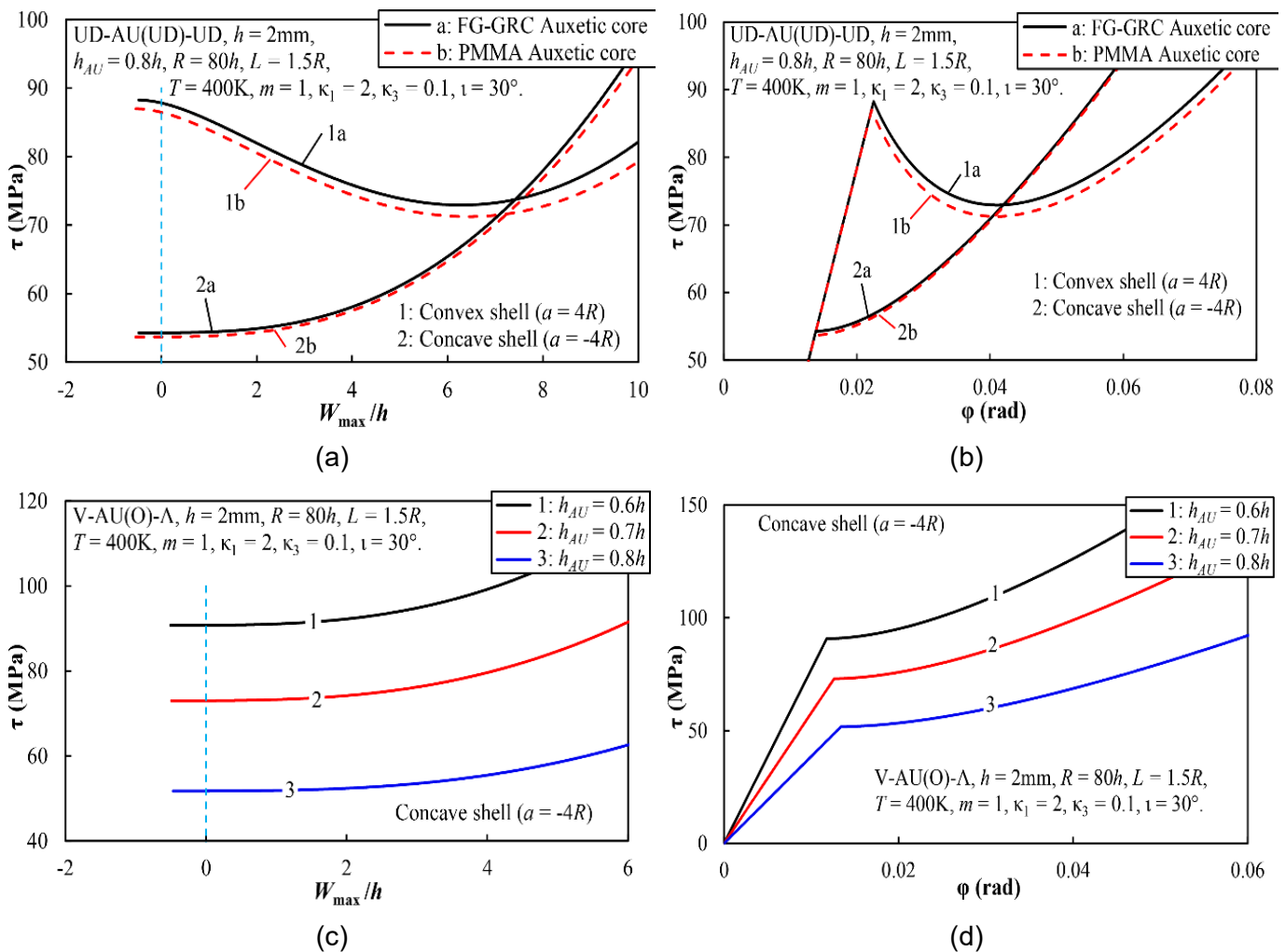


Fig. 2. Effects of auxetic core and core thickness on the postbuckling behavior of shells. (a) Effects of auxetic core ($\tau - W_{\max}/h$ curve); (b) Effects of auxetic core ($\tau - \phi$ curve); (c) Effects of core thickness ($\tau - W_{\max}/h$ curve); (d) Effects of core thickness ($\tau - \phi$ curve)

Fig. 2 illustrates the effects of auxetic core material and core thickness on the postbuckling behavior of FG-GRC shell structures. As shown in

Fig. 2(a), the $\tau - W_{\max}/h$ curves indicate that the shell with an FG-GRC auxetic core exhibits slightly higher postbuckling resistance than that with a

PMMA auxetic core for both convex and concave configurations. The result confirms that replacing the conventional PMMA core with an FG-GRC auxetic core enhances the stiffness of the sandwich shell and improves its torsional load-carrying capacity. Fig. 2(a) also shows a clear difference between convex and concave shells. The convex shell exhibits a pronounced snap-through-type postbuckling response, characterized by an initial peak load followed by a descending branch and then a gradual recovery in load capacity as the deflection increases. In contrast,

the concave shell does not display a significant load drop and follows a smoother monotonic path. The same tendency is observed in the torsion-twist angle curves in Fig. 2(b). For a given twist angle, the FG-GRC auxetic-core shell generally sustains a slightly larger torsional load than the PMMA-core shell. In addition, the convex shell again shows a non-monotonic postbuckling path with a distinct reduction after the upper critical point, whereas the concave shell exhibits a more stable and gradual increase of torsional load with increasing twist angle.

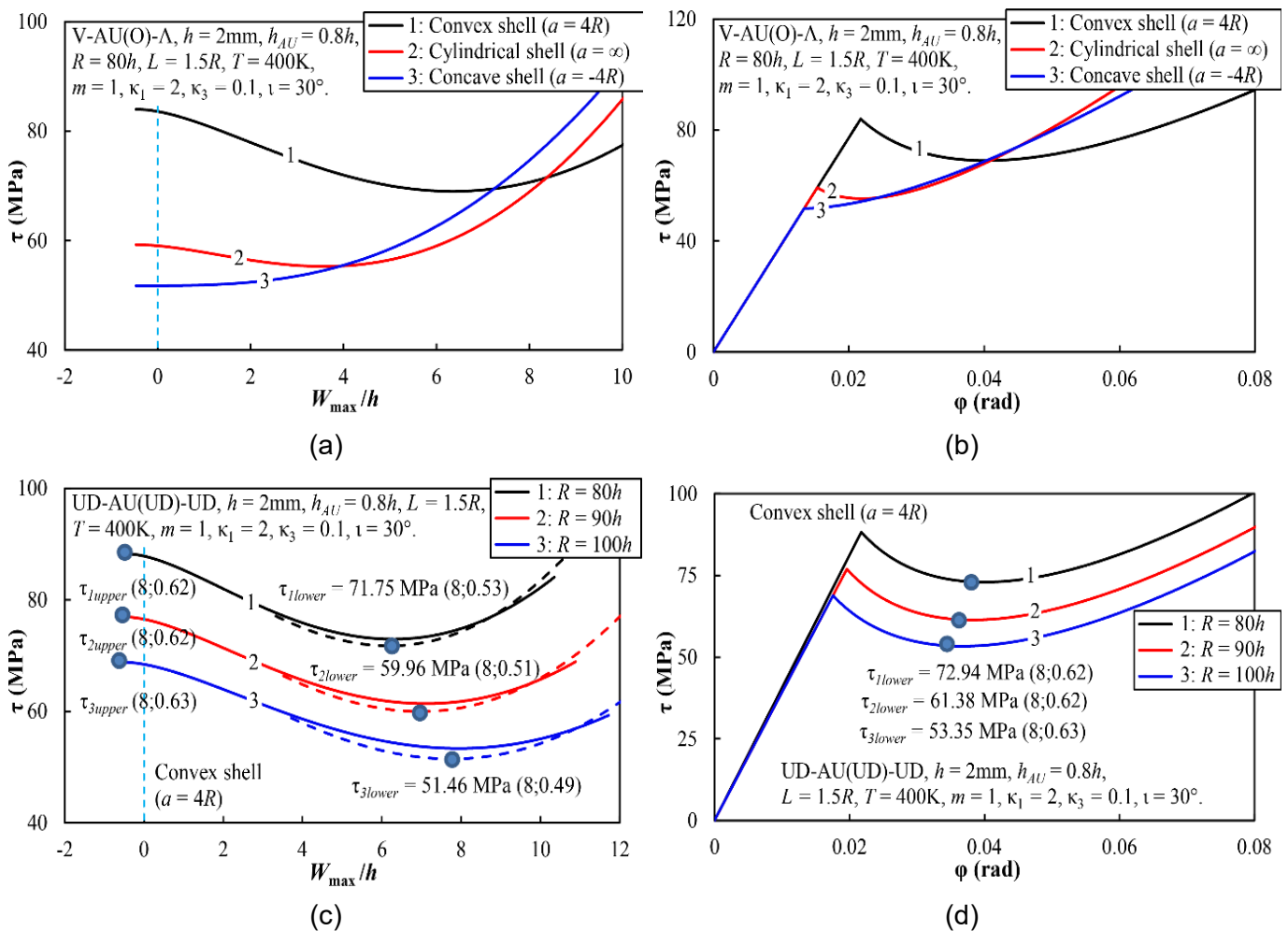


Fig. 3. Effects of shell type and shell radius on the postbuckling behavior of shells. (a) Effects of shell type ($\tau - W_{\max}/h$ curve); (b) Effects of shell type ($\tau - \phi$ curve); (c) Effects of shell radius ($\tau - W_{\max}/h$ curve); (d) Effects of shell radius ($\tau - \phi$ curve)

Figs. 2(c) and 2(d) present the influence of core thickness. It is evident that increasing the auxetic core thickness leads to a noticeable decrease in both the torsional buckling load and

the postbuckling strength. In the $\tau - W_{\max}/h$ curves of Fig. 2(c), thicker cores shift the postbuckling paths downward, indicating reduced resistance against torsional deformation. A similar

trend is observed in the $\tau - \varphi$ curves of Fig. 2(d), where larger values of h_{AU} correspond to lower torsional loads throughout the postbuckling range.

Fig. 3 presents the effects of shell type and shell radius on the postbuckling behavior of FG-GRC toroidal shell segments with an auxetic core through the $\tau - W_{\max}/h$ and $\tau - \varphi$ relationships. As shown in Fig. 3(a), the shell type has a pronounced influence on the torsion–deflection response. The convex shell exhibits the highest load-carrying

capacity in the prebuckling and early postbuckling stages, together with a clear snap-through-type response characterized by an upper critical point followed by a descending branch. In contrast, the cylindrical and concave shells carry lower torsional loads, and their responses are smoother. In particular, the concave shell initially has the lowest resistance, but its postbuckling curve rises more rapidly at larger deflections and eventually approaches or even exceeds the cylindrical shell response.

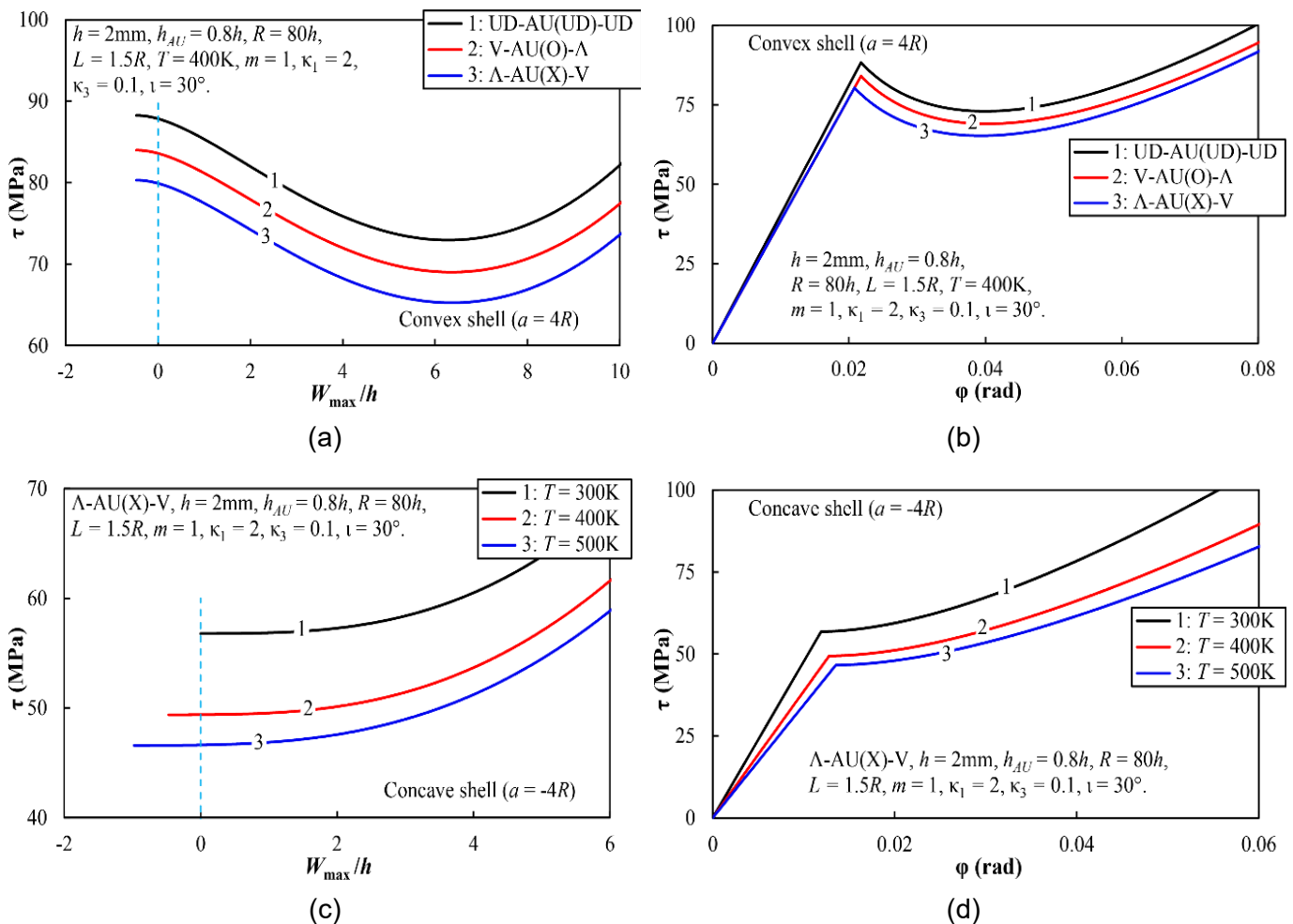


Fig. 4. Effects of graphene distribution and temperature on the postbuckling behavior of shells. (a) Effects of graphene distribution ($\tau - W_{\max}/h$ curve); (b) Effects of graphene distribution ($\tau - \varphi$ curve); (c) Effects of temperature ($\tau - W_{\max}/h$ curve); (d) Effects of temperature ($\tau - \varphi$ curve)

A similar tendency is observed in Fig. 3(b) for the $\tau - \varphi$ curves. The convex shell again shows the most pronounced non-monotonic behavior, with a distinct reduction in torsional load after the upper critical point, confirming its snap-through instability

under torsion. The concave shell starts from a lower load level, but its stiffness recovery in the postbuckling regime is relatively fast, resulting in a steeply increasing branch at larger twist angles.

Figs. 3(c) and 3(d) illustrate the influence of

the shell radius on the postbuckling response of the convex shell. It is evident from Fig. 3(c) that increasing the radius-to-thickness ratio reduces both the upper and lower critical torsional loads. Moreover, the postbuckling curves shift downward as R/h increases, indicating a loss of torsional stiffness and stability for thinner shells. This behavior is physically reasonable because a larger R/h ratio corresponds to a more slender shell, which is more susceptible to instability under torsion. The same effect can be observed in Fig. 3(d) for the $\tau - \varphi$ relation. As the shell radius increases, the entire torsion–twist angle curve moves downward, indicating a reduction in torsional resistance throughout the postbuckling regime. Although all three cases preserve the same qualitative snap-through pattern, the shell with $R/h = 80$ consistently sustains the highest torsional loads, whereas the shell with $R/h = 100$ shows the lowest.

Fig. 4 presents the effects of graphene distribution and temperature on the postbuckling behavior of FG-GRC auxetic-core toroidal shell segments.

As shown in Fig. 4(a), the graphene distribution pattern has a clear influence on the $\tau - W_{\max}/h$ response of the convex shell. Among the three investigated patterns, the UD-AU(UD)-UD configuration consistently provides the highest torsional load over the entire postbuckling range, whereas the Λ -AU(X)-V pattern gives the lowest values, with V-AU(O)- Λ lying in between. In addition, all three curves exhibit a similar snap-through-type response, characterized by an initial upper critical point, a descending branch, and a subsequent stiffness recovery at larger deflections.

A similar trend is observed in Fig. 4(b) for the $\tau - \varphi$ relation of the convex shell. The three torsion–twist angle curves preserve the same qualitative pattern, but the shell with UD-AU(UD)-

UD distribution sustains the highest torsional load for a given twist angle, whereas the Λ -AU(X)-V distribution leads to the lowest torsional resistance.

The influence of temperature is illustrated in Figs. 4(c) and 4(d) for the concave shell with the Λ -AU(X)-V distribution. It is evident that increasing temperature shifts the postbuckling curves downward in both the $\tau - W_{\max}/h$ and $\tau - \varphi$ diagrams. The reduction in torsional buckling and postbuckling strength with increasing temperature can be attributed to the degradation of the effective material stiffness of FG-GRC constituents under thermal exposure. Nevertheless, although temperature strongly affects the load level, the general shape of the postbuckling curves is preserved, indicating that thermal effects mainly modify the magnitude of the response rather than the fundamental postbuckling mechanism.

6. Conclusions

In this study, an analytical Ritz-based formulation was developed to investigate the thermo-mechanical nonlinear torsional buckling and postbuckling behavior of FG-GRC toroidal shell segments with an auxetic core. Unlike conventional sandwich configurations employing polymer-based cores, the present model considered a fully FG-GRC structure in which both the facesheets and the auxetic honeycomb core were made of functionally graded graphene-reinforced composite. The effective material properties of the FG-GRC layers were evaluated using the extended Halpin-Tsai model, while the auxetic core was homogenized as an equivalent orthotropic medium. Based on the nonlinear Donnell shell theory with von Kármán geometric nonlinearity and the Stein-McElman approximation, closed-form expressions were derived for the critical torsional buckling load, maximum deflection, and twist angle. From the obtained results, the following main conclusions can be drawn:

The shell curvature has a pronounced influence on the torsional buckling behavior. Convex toroidal shell segments possess the highest critical torsional buckling loads and exhibit a clear snap-through-type postbuckling response, whereas concave shells show lower torsional resistance and a smoother postbuckling path.

The use of an FG-GRC auxetic core leads to slightly higher critical torsional buckling loads and improved postbuckling resistance compared with the PMMA auxetic core.

Among the investigated cases, the UD-AU(UD)-UD pattern consistently provides the highest buckling and postbuckling loads, whereas the Λ -AU(X)-V pattern yields the lowest values.

Increasing temperature causes a marked reduction in the critical torsional buckling load and postbuckling strength for all considered shell configurations.

A thicker core significantly decreases both the critical load and the postbuckling load-carrying capacity for the considered shell configurations.

Declaration of Generative AI

During the preparation of this work, the authors did not use Generative AI and AI-assisted technology tools in order to generate the content of the manuscript.

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