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Nonlinear transient response of porous FGM sandwich beams with elastically restrained ends in a thermal environment

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Abstract: This study executes a semi–analytical investigation on the combined effects of pores, tangential ends constraints, initial geometric imperfection, foundation interaction, and high temperatures on the nonlinear transient response of functionally graded material (FGM) sandwich beams subjected to suddenly applied uniform transverse loads in thermal environments. Two types of sandwich beams corresponding to FGM face sheets and core layer are examined. The pores are distributed into FGM layers according to even and uneven types. The effective properties of porous FGM are estimated employing a modified rule of mixture. The motion equations are formulated on the basis of the first order shear deformation beam theory taking into consideration von Kármán nonlinear terms, initial geometric imperfection and interaction from elastic foundations. The derived equations are treated by means of analytical solutions along with Galerkin method to attain a time–dependent nonlinear ordinary differential equation. This differential equation is numerically treated by means of fourth–order Runge–Kutta scheme to trace nonlinear displacement–transient time paths. Parametric studies are performed to evaluate diverse effects on the nonlinear transient response curves of FGM sandwich beams. The results indicate that the pores and foundations have detrimental and beneficial effects on the dynamical bending resistance capability of sandwich beams, respectively. The study also finds that the tangential end restraints make the dynamical deflection of beams exposed to thermal environments deeper. Besides, the initial geometric imperfection decreases and increases the dynamic deflection of the beams placed at room and elevated temperatures, respectively.

Keywords: Porous FGM sandwich beam; Nonlinear transient response; Elastic end restraint; Winkler-Pasternak foundation; Geometric imperfection.

1. Introduction

Owing to advanced characteristics inheriting from superior features of constitutive components, functionally graded material (FGM) overcomes issues encountered in conventional fiber reinforced composites such as high stress concentration and debonding. Due to smooth and continuous gradation of volume percentages of ceramic and metal through one or two directions, FGM owns high stiffness, ductility, effective performance temperature resistance capability and prominent structural integrity. Therefore, FGMs are widely applied in diverse fields such as temperature withstanding components, aerospace structures and nuclear plants. The static and dynamic responses of FGM beams (FGMBs) were studied in numerous papers. Wattanasakulpong et al. [1] used an improved version of Reddy third order beam theory (TSDBT) and Ritz technique to treat the linear buckling and free vibration of FGMBs under thermal loads. Linear free vibration of FGMBs with various end conditions has been perused by Pradhan and Chakraverty [2] employing Rayleigh–Ritz solution on the basis of the Euler–Bernoulli classical beam theory (CBT) and the Timoshenko's first order shear deformation beam theory (FSDBT). Lagrange multipliers method on the context of different higher–order theories was adopted in paper of Simsek [3] for the linear vibration analysis of FGMBs. Azadi [4] taken up the finite element method (FEM) based on the CBT to deal with the linear free and forced vibrations of FGMBs with different ends constraints. An exact solution treatment based on the FSDBT was utilized in work of Rahimi et al. [5] for the postbuckling behavior and linear free vibration analyses of FGMBs under various end conditions. Thai and Vo [6] executed an analytical study on the linear static and vibration behaviors of FGMBs taking advantage of various versions of the TSDBT. Wang and Li [7] applied a shooting method within the framework of Levinson beam theory for the linear vibration analysis of FGMBs. The CBT–based FEM was taken up in the study of Alshorbagy et al. [8] for the linear free vibration issue of FGMBs. Karamanli [9] used displacement and rotation solutions in polynomial forms and the FSDBT to solve the linear vibration problem of FGMBs. Rayleigh–Ritz method and experimental comparison were employed in study of Pang et al. [10] for the linear free and forced vibration analyses of FGMBs with general restraints at ends.

Nonlinear analyses of the vibrational and buckling behaviors of FGMBs were performed in studies [11-15]. Fallah and Aghdam [11] used the CBT and variational method to investigate the nonlinear free vibration and postbuckling of thin FGMBs resting on nonlinear elastic foundations. The effects of piezoelectric actuators on the nonlinear free vibration and buckling of FGMBs with clamped ends have been analyzed by Fu et al. [12] employing the CBT and elliptic function solution. The variational iteration method based on the CBT was adopted in work of Yaghoobi and Torabi [13] for the postbuckling and nonlinear vibration analyses of FGM beams supported by nonlinear elastic foundations. Zhang [14] taken up the TSDBT–based Ritz method to deal with the thermal postbuckling and free vibrational response of FGMBs taking into consideration the physical neutral surface. Shen and Wang [15] dealt with the nonlinear analyses of the static response, free vibration and thermal postbuckling of FGMBs resting on elastic foundations taking up TSDBT in conjunction with a two-step perturbation technique. The CBT–based multiple scale method was applied in work of Shooshtari and Rafiee [16] for the nonlinear forced vibration problem of FGMBs. Hemmatnezhad et al. [17] taken advantage of the FSDBT–based FEM to examine the nonlinear free vibration of FGMBs under various end conditions. The influence of nonlinear foundations on the nonlinear free and forced vibrational behaviors of FGMBs has been analyzed by Kanani et al. [18] utilizing variational iteration technique on the basis of the CBT. The nonlinear free vibration analysis of shear deformable FGMBs was executed by Xie et al. [19] making use of Ritz method within the context of various shear deformation theories. The nonlinear vibrational and dynamic responses of thick FGMBs were researched by Xie et al. [20] adopting an improved TSDBT along with a direct numerical integration technique.

Owing to prominent features such as high ratio of flexural rigidity to weight along with effective sound and thermal absorption, sandwich components are extensively applied in diverse structures. Zenkour [21] employed Navier–type series solution on the basis of sinusoidal version of shear deformation theory to explore the linear buckling and free vibration of FGM sandwich rectangular plates with all simply supported edges. By utilizing a meshless method in conjunction with a quasi-3D version of shear deformation theory, Neves et al. [22] perused the linear bending, free vibration and buckling behaviors of FGM sandwich plates under different boundary conditions. Based on a refined shear deformation theory, the linear static, buckling and free vibrational behaviors of FGM sandwich plates were elaborated in studies of Meiche et al. [23] and Bessaim et al. [24] using analytical solutions. Based on a higher order shear deformation theory in conjunction with an asymptotic solution, Shen and coauthors [25, 26] presented the postbuckling and nonlinear dynamical response analyses of FGM sandwich rectangular plates supported by elastic foundations in thermal environments. The nonlinear buckling of FGM sandwich plates and curved panels subjected to elevated temperature and thermomechanical loads have been examined by Tung [27, 28] using analytical solutions on the basis of the shear deformation theories. Iurlaro et al. [29] used a refined version of zigzag theory for the linear static and free vibrational behavior analysis of FGM sandwich plates. The FSDBT–based general Fourier solution was utilized in work of Su et al. [30] for determining the natural frequencies of sandwich FGMBs under arbitrary boundary conditions resting on elastic foundations.

Inasmuch as the pores can be created in FGM during the manufacturing process, it is necessary to assess the influences of porosity on the static and dynamic behaviors of FGM structures. Wattanasakulpong and Ungbhakorn [31] adopted the differential transformation technique based on the CBT to deal with the linear and nonlinear vibration responses of FGMBs with porosity and elastically constrained ends. The FSDBT–based collocation technique was employed in study of Wattanasakulpong and Chaikittiratana [32] for the linear free vibration analysis of FGMBs including porosities. Based on the FSDBT and a semi–analytical approach, Thinh and coworkers [33, 34] investigated the thermal postbuckling and nonlinear free vibration of FGMBs with porosities and flexibly constrained ends. The TSDBT–based Navier–type solution was utilized in work of Ebrahimi and Jafari [35] for the linear free vibrational behavior analysis of FGMBs with porosity. The impacts of pores on the linear bending and free vibrational problems of FGMBs with various end conditions have been treated in numerical investigations of Zghal et al. [36] and Akbas [37], respectively, employing the FEM on the basis of shear deformation theories. Turan et al. [38] used some various approaches to look at the linear free vibration and buckling problems of FGMBs with porosity. Long and Tung [39] considered the influences of porosity and preexistent compressive load on the nonlinear free vibration of FGM sandwich rectangular plates with restrained unloaded edges. The effects of pores and tangential boundary restraints on the thermal postbuckling and nonlinear transient response of FGM single–layered and sandwich plates were addressed in studies [40, 41] using analytical solution on the context of the FSDPT. Recently, Thinh and Tung [42, 43] presented semi–analytical investigations on the nonlinear transient response of FGM circular plates, doubly curved panels and spherical caps with porosities and elastically constrained boundary edges. It is evident that the nonlinear dynamical response of FGM sandwich beams with porosity has not been investigated.

This paper presents a semi–analytical approach to examine the nonlinear transient response of FGM sandwich beams with porosity exposed to thermal environments and subjected to suddenly applied uniform transverse loads. Two ends of beams are assumed to be simply supported and elastically constrained against axial displacement. Two sandwich beam models with FGM face sheets and core layer are studied. Pores are distributed into FGM layers via even and uneven patterns. The effective properties of porous

FGM are estimated employing a modified rule of mixture. Motion equations are established based on the FSDBT incorporating von Kármán nonlinearity, initial geometric imperfection and Winkler–Pasternak type foundation interaction. The established equations are solved employing analytical solutions along with Galerkin method. Fourth–order Runge–Kutta integration scheme is applied to trace the dynamic deflection–transient time paths. Parametric studies are executed to examine various effects on the nonlinear dynamic response of porous sandwich beams. From the attained results, insightful discussions are given and valuable remarks are suggested.

2. Porous FGM sandwich beams

This work examines a sandwich beam of length L, unit width b, and total thickness h constructed from homogeneous and FGM layers. The beam is rested on an elastic foundation and defined in a Cartesian coordinate system xyz in which the origin is placed on the mid-plane at an end, x and z are length and thickness coordinates, respectively, as plotted in Fig. 1. The sandwich beam is fabricated from a thicker core layer and two face sheets (i.e. skin layers) in which the thickness of each skin layer is h_f. It is assumed that the core layer and face sheets are perfectly bonded. Two sandwich beam models owning FGM face sheets and core layer referred to herein as sandwich beams of types A and B, respectively, will be studied in this work.

2.1. Type A of porous sandwich beam: homogeneous core and functionally graded face layers

In this model, the sandwich beam is constructed from metal homogeneous core layer and FGM face sheets, as illustrated in Fig. 2. The volume fraction of ceramic constituent is continuously decreased from the top and bottom surfaces to two interfaces of the sandwich beam. This type of FGM sandwich beam is symmetric about middle plane and the volume fraction of ceramic component is evaluated as the following

$$V_c = \left(\frac{z-h_1}{h_0-h_1}\right)^N,\ h_0 \le z \le h_1,\ \text{top face sheet}$$

$$V_c = 0,\ h_1 \le z \le h_2,\ \text{core layer} \quad (1)$$

$$V_c = \left(\frac{z-h_2}{h_3-h_2}\right)^N,\ h_2 \le z \le h_3,\ \text{bottom face sheet}$$

Here h0 = -h/2, h1 = -h/2 + h_f, h2 = h/2 - h_f and h3 = h/2; N is a non-negative volume fraction index.

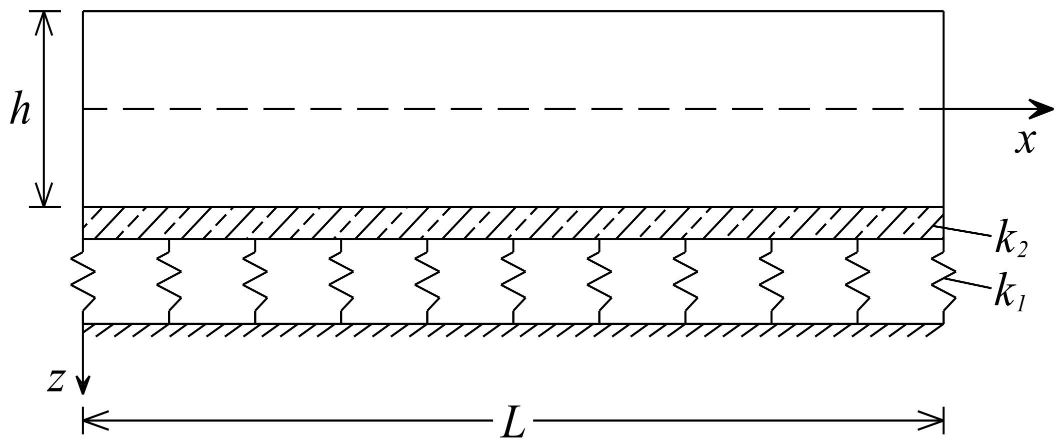


Fig. 1. Geometry and coordinates of a beam resting on an elastic foundation

2.2. Type B of porous sandwich beam: FGM core layer and homogeneous face sheets

In this type of sandwich configuration, the beam is constructed from homogeneous face sheets and a FGM core layer, as sketched in Fig. 3. Specifically, the upper and lower face layers are homogeneous metal and ceramic, respectively, whereas the volume fraction of ceramic component in the FGM core is gradually increased from upper interface to lower interface. This configuration of the sandwich beam is asymmetrical about middle surface and the volume fraction of ceramic component is determined as the following:

$$V_c = 0,\ h_0 \le z \le h_1,\ \text{top face sheet}$$

$$V_c = \left(\frac{z-h_1}{h_2-h_1}\right)^N,\ h_1 \le z \le h_2,\ \text{core layer} \qquad (2)$$

$$V_c = 1,\ h_2 \le z \le h_3\ ,\ \text{bottom face sheet}$$

For both types of sandwich beam, the volume fraction of metal constituent is evaluated by formula $V_m = 1 - V_c$.

During the fabrication of FGM, pores can arise and affect the properties of this composite. Because of the existence of pores, the effective properties P_{eff} of porous FGM may be estimated by means of a modified version of mixture rule as the following [31].

$$P_{eff} = P_c\left(V_c - \frac{\xi}{2}\right) + P_m\left(V_m - \frac{\xi}{2}\right) \qquad (3)$$

where P stands for a specific property such as elastic modulus E, thermal expansion coefficient (TEC) α and mass density ρ. Herein the c and m subscripts imply the ceramic and metal constituents, respectively. In the above Eq. (3), ξ is a small value ($0 \le \xi \ll 1$) known as the total volume fraction of pores. If ξ is equal to zero, FGM is absent from pores and known as perfect or non-porous FGM. In the present study, the pores are considered to distribute inside FGM according to even and uneven patterns. For the even type of distribution, the density of pores is the same for arbitrary position in the FGM and the effective properties of porous FGM are determined as [31]

$$P_{eff} = (P_c - P_m)V_c + P_m - \frac{\xi}{2}(P_c + P_m) \qquad (4)$$

For the uneven type of porosity distribution, the density of pores is non-uniform in the FGM. For the type A of sandwich beam, the density of pores is gradually increased from two surfaces to the interfaces, as sketched in Fig. 2. For the type B of sandwich beam, the density of pores is linearly increased from two interfaces to the middle surface of the beam, as depicted in Fig. 3. Accordingly, the density of pores is the densest at two interfaces and the middle surface of the sandwich beams of types A and B, respectively. The effective properties of porous FGM with uneven porosity are sought by the following expression.

$$P_{eff} = (P_c - P_m)V_c + P_m - \frac{\xi}{2}(P_c + P_m)f(z) \qquad (5)$$

For the A sandwich beam, $f(z) = \frac{z-h_0}{h_1-h_0}$, 0, $\frac{z-h_3}{h_2-h_3}$ corresponding with $h_0 \le z \le h_1$, $h_1 \le z \le h_2$, $h_2 \le z \le h_3$, respectively. For the sandwich beam of type B, $f(z) = \left(1 - 2\frac{|z|}{h_2-h_1}\right)$ at core layer ($h_1 \le z \le h_2$) and $f(z) = 0$ at two face sheets ($h_0 \le z \le h_1$ and $h_2 \le z \le h_3$).

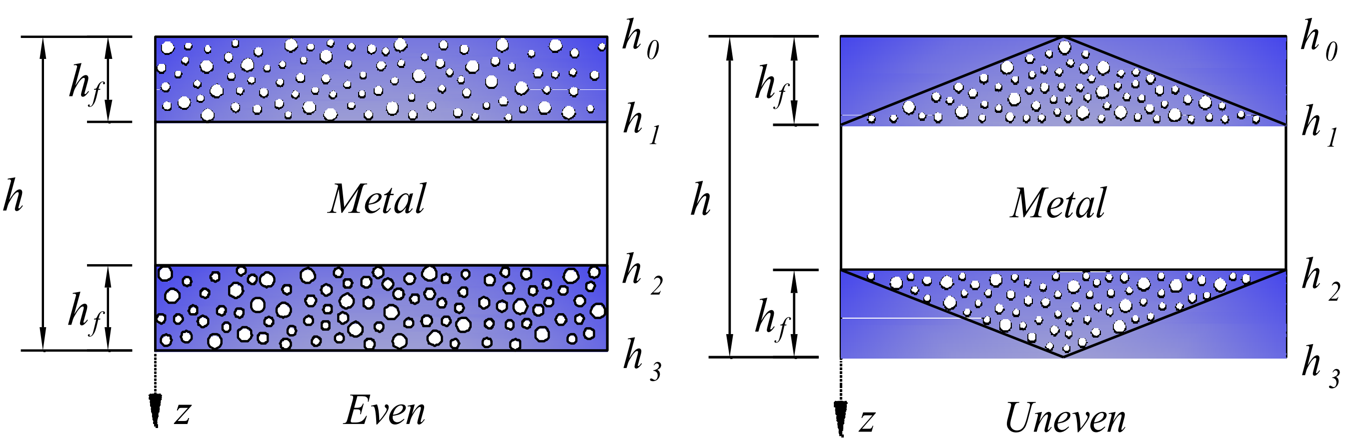


Fig. 2. FGM/Metal/FGM sandwich model with even and uneven types of porosity distribution

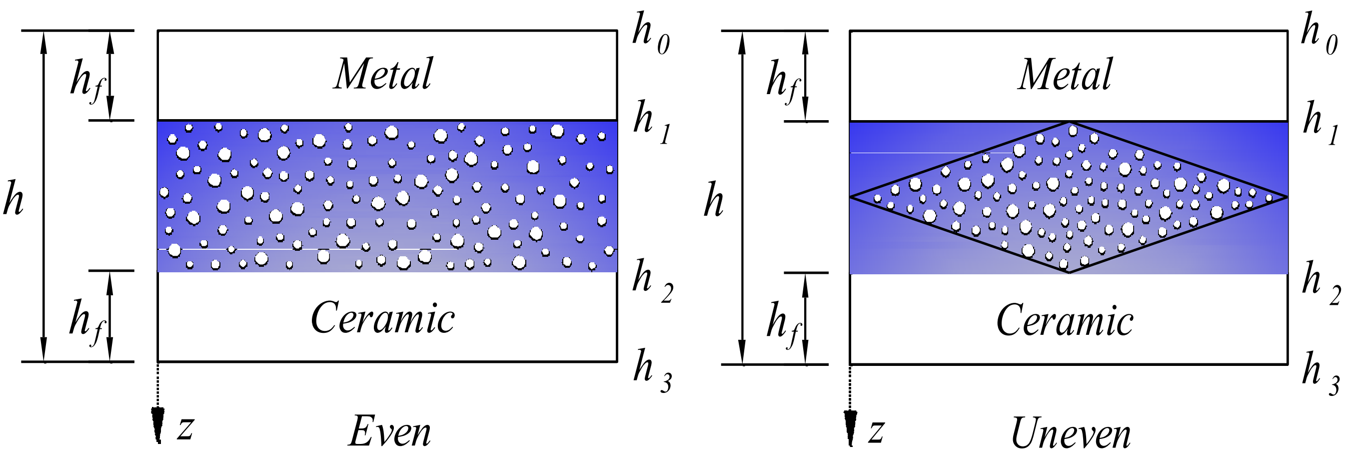


Fig. 3. Metal/FGM/Ceramic sandwich model with even and uneven types of porosity distribution

Because of weak dependence on position, temperature and porosity, Poisson's ratio ν of porous FGM is assumed to be constant in the present work.

3. Mathematical formulations

In this study the sandwich beams are considered to be of moderate thickness and imperfect geometry. Motion equations are established on the basis of the FSDBT. Based on the FSDBT, axial displacement $\bar{u}(x,z,t)$ and transverse displacement (i.e. deflection) $\bar{w}(x,z,t)$ of an arbitrary point in the beam are expressed in the following [44].

$$\begin{aligned}\bar{u}(x,z,t) &= u(x,t) + z\phi(x,t)\\ \bar{w}(x,z,t) &= w(x,t)\end{aligned} \tag{6}$$

where t is time. In the above, u and w are axial and transverse displacements of a corresponding point on the mid-surface, respectively, and ϕ is the rotation of a transverse normal about the y axis. Axial strain ε_{xx} and transverse shear deformation γ_{xz} at an arbitrary point are defined as follows [44]

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi}{\partial x},\ \gamma_{xz} = \phi + \frac{\partial w}{\partial x} \tag{7}$$

The components of stresses are sought by means of constitutive relations as

$$\sigma_{xx} = E\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi}{\partial x} - \alpha\Delta T\right] \tag{8}$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)}\left(\phi + \frac{\partial w}{\partial x}\right)$$

where $\Delta T = T - T_0$ is temperature difference uniformly raised from a room value T_0 to a higher value T. In this study, it is assumed that the uniform temperature rise induces thermal stress in the sandwich beam prior to application of transverse mechanical load. Based on this assumption, thermal load is seen as a pre-existent static load. The force resultant N_x, the shear force resultant Q_x and the moment resultant M_x are determined as follows [44].

$$\begin{aligned}(N_x, M_x) &= \int_{-h/2}^{h/2} \sigma_{xx}(1,z)dz\\ Q_x &= K_S \int_{-h/2}^{h/2} \sigma_{xz}dz\end{aligned} \tag{9}$$

where $K_S = 5/6$ is selected for the value of shear correction coefficient. Substituting Eq. (8) into Eq. (9) gives the following relations of the force and moment resultants

$$N_x = A_{11}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) + B_{11}\frac{\partial \phi}{\partial x} - N^T \tag{10a}$$

$$M_x = B_{11}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) + D_{11}\frac{\partial \phi}{\partial x} - M^T \tag{10b}$$

$$Q_x = K_S A_{55}\left(\phi + \frac{\partial w}{\partial x}\right) \tag{10c}$$

where

$$\left(A_{11},B_{11},D_{11}\right)=\int_{-h/2}^{h/2}E\left(1,z,z^2\right)dz$$

$$A_{55}=\frac{A_{11}}{2(1+\nu)} \tag{11}$$

$$\left(N^T,M^T\right)=\int_{-h/2}^{h/2}E\alpha\Delta T\left(1,z\right)dz$$

This study assumes that axial and rotatory inertias are marginal and removed. Based on this assumption and the FSDBT, the motion equations of FGM sandwich beams with imperfect geometry resting on an elastic foundation are the following [34, 44].

$$\frac{\partial N_x}{\partial x}=0 \tag{12a}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial}{\partial x}\left(N_x\frac{\partial\left(w+w^*\right)}{\partial x}\right)+q-q_f=I_1\frac{\partial^2 w}{\partial t^2} \tag{12b}$$

$$\frac{\partial M_x}{\partial x}-Q_x=0 \tag{12c}$$

with $w^*(x)$ is a known function describing initial geometric imperfection and q is uniform transverse load suddenly applied on the top plane of the FGM sandwich beam. The considered transverse load type is usually seen in the practical situations of diverse components such as rails and barriers. This load is perpendicular to the longitudinal axis of the sandwich beam and applied instantaneously rather than gradually. In the above, q_f is interaction force from the foundation and I_1 is the mass inertia moment defined as

$$I_1=\int_{-h/2}^{h/2}\rho(z)dz \tag{13}$$

In this study, interaction force from elastic foundation is expressed according to Winkler–Pasternak model as

$$q_f=k_1w-k_2\frac{\partial^2 w}{\partial x^2} \tag{14}$$

in which k_1 and k_2 are stiffness parameters of a Winkler–Pasternak foundation.

It is evident from Eq. (12a) that $N_x=N_{x0}$ is independent of axial coordinate x. Using this relation and Eq. (12c) into (12b) yields

$$\frac{\partial^2 M_x}{\partial x^2}+N_{x0}\frac{\partial^2(w+w^*)}{\partial x^2}+q-k_1w+k_2\frac{\partial^2 w}{\partial x^2}=I_1\frac{\partial^2 w}{\partial t^2} \tag{15}$$

It is deduced from Eqs. (10a) and (10b) that

$$M_x=\frac{B_{11}}{A_{11}}N_x+\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\frac{\partial\phi}{\partial x}+\frac{B_{11}}{A_{11}}N^T-M^T \tag{16}$$

By substituting the above relation into Eq. (15), the motion equation of the sandwich beam is written in the form

$$\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\frac{\partial^3\phi}{\partial x^3}+N_{x0}\frac{\partial^2}{\partial x^2}\left(w+w^*\right)+q-k_1w+k_2\frac{\partial^2 w}{\partial x^2}=I_1\frac{\partial^2 w}{\partial t^2} \tag{17}$$

Introducing Eqs. (10c) and (16) into Eq. (12c) yields the following

$$\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\frac{\partial^2\phi}{\partial x^2}-K_SA_{55}\left(\phi+\frac{\partial w}{\partial x}\right)=0 \tag{18}$$

This study considers FGM sandwich beams with simply supported ends including elastic constraint against axial displacement. The corresponding end conditions are written as

$$w=0,\ M_x=0,\ N_x=N_{x0}$$

$$\text{at } x=0 \text{ and } x=L \tag{19}$$

Reactive force resultant at the beam ends is related to average end-shortening displacement as described in previous works [34, 45-48].

$$N_{x0}=-\frac{c}{L}\int_0^L\frac{\partial u}{\partial x}dx \tag{20}$$

where c is stiffness parameter of axial restraints of ends. The values of $c=0$, $c\to\infty$ and $0<c<\infty$ describe the movable, immovable and partially movable conditions of ends, respectively. By determining $\partial u/\partial x$ from Eq. (10a) and including the effect of geometric imperfection, Eq. (20) can be rewritten in the following form

$$N_{x0}=-\frac{c}{L}\int_0^L\left[\begin{array}{l}\frac{1}{A_{11}}\left(N_{x0}+N^T-B_{11}\frac{\partial\phi}{\partial x}\right)\\ -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{\partial w}{\partial x}\frac{\partial w^*}{\partial x}\end{array}\right]dx \quad (21)$$

4. Solution procedure

The end conditions are fulfilled by means of the following analytical solutions [34, 49]

$$w(x,t)=W(t)\sin\frac{m\pi x}{L}$$

$$w^*(x)=\mu h\sin\frac{m\pi x}{L} \quad (22)$$

$$\phi(x,t)=\Phi(t)\cos\frac{m\pi x}{L}$$

where W and Φ are amplitudes of transverse displacement and rotation, respectively, μ is the size of geometric imperfection and m is a positive integer representing the vibration mode. By substituting Eq. (22) into Eq. (18), we receive

$$\Phi=-\bar{\Phi}\bar{W} \quad (23)$$

where

$$\bar{\Phi}=\frac{K_S\bar{A}_{11}\bar{A}_{55}m\pi L_h}{m^2\pi^2\left(\bar{A}_{11}\bar{D}_{11}-\bar{B}_{11}^2\right)+K_S\bar{A}_{11}\bar{A}_{55}L_h^2},$$

$$\bar{W}=\frac{W}{h} \quad (24)$$

in which

$$\left(\bar{A}_{11},\bar{A}_{55},L_h\right)=\frac{1}{h}(A_{11},A_{55},L)$$

$$\bar{B}_{11}=\frac{B_{11}}{h^2},\ \bar{D}_{11}=\frac{D_{11}}{h^3} \quad (25)$$

By introducing the solutions (22) into the motion equation (17) and taking up Galerkin method, we receive the following equation

$$I_1h\frac{d^2\bar{W}}{dt^2}+m^3\pi^3\frac{\bar{A}_{11}\bar{D}_{11}-\bar{B}_{11}^2}{\bar{A}_{11}L_h^3}\bar{\Phi}\bar{W}$$

$$+\frac{E_m^0}{12L_h^4}\left(K_1+m^2\pi^2K_2\right)\bar{W}+\bar{N}_{x0}\frac{m^2\pi^2}{L_h^2}\left(\bar{W}+\mu\right)$$

$$=\frac{4\gamma_m}{m\pi}q \quad (26)$$

where

$$\bar{N}_{x0}=\frac{N_{x0}}{h}$$

$$\left(K_1,K_2\right)=\frac{12L^2}{E_m^0h^3}\left(k_1L^2,k_2\right) \quad (27)$$

$$\gamma_m=\frac{1}{2}\left[1-(-1)^m\right]$$

in which E_m^0 is value of E_m calculated at $T_0=300$ K. Introducing the solutions (22) into Eq. (21) leads to the result

$$\bar{N}_{x0}=-\lambda G\Delta T+2\lambda\gamma_m\frac{\bar{B}_{11}}{L_h}\bar{\Phi}\bar{W}$$

$$+\lambda\bar{A}_{11}\frac{m^2\pi^2}{4L_h^2}\bar{W}\left(\bar{W}+2\mu\right) \quad (28)$$

in which

$$\lambda=\frac{c}{c+A_{11}},\quad G=\frac{1}{h}\int_{-h/2}^{h/2}E\alpha dz. \quad (29)$$

Above defined dimensionless value λ will be used to assess the degree of tangential restraints of boundary ends in parametric studies. It is realized that freely movable, fully immovable and partially movable conditions of ends will be measured by values of $\lambda=0$, $\lambda=1$ and $0<\lambda<1$, respectively. Substitution of $\bar{N}_{x0}$ into Eqs. (26) leads to

$$I_1h\frac{d^2\bar{W}}{dt^2}+a_{11}\bar{W}+a_{21}\bar{W}\left(\bar{W}+\mu\right) \quad (30)$$

$$+a_{31}\bar{W}\left(\bar{W}+\mu\right)\left(\bar{W}+2\mu\right)-a_{41}\left(\bar{W}+\mu\right)\Delta T=\frac{4\gamma_m}{m\pi}q$$

where

$$a_{11}=m^3\pi^3\frac{\bar{A}_{11}\bar{D}_{11}-\bar{B}_{11}^2}{\bar{A}_{11}L_h^3}\bar{\Phi}+\frac{E_m^0}{12L_h^4}\left(K_1+m^2\pi^2K_2\right)$$

$$a_{21}=2\lambda\gamma_m\frac{m^2\pi^2\bar{B}_{11}}{L_h^3}\bar{\Phi} \quad (31)$$

$$a_{31}=\lambda\frac{m^4\pi^4\bar{A}_{11}}{4L_h^4},\ a_{41}=\lambda\frac{m^2\pi^2}{L_h^2}G$$

Table 1. Comparisons of dimensionless fundamental natural frequencies $\bar{\omega}$ of simply supported Al/Al_2O_3 beams with immovable ends and without foundation interaction ($L/h = 20$, $\Delta T = 0$, $N = 2$, $m = 1, \lambda = 1$).

Source	Even			Uneven		
	$\xi = 0$	$\xi = 0.1$	$\xi = 0.2$	$\xi = 0$	$\xi = 0.1$	$\xi = 0.2$
Turan et al. [38]	3.8368	3.6340	3.3128	3.8368	3.8235	3.8017
Hadji et al. [50]	3.8361	3.6335	3.3123	3.8361	3.8226	3.8004
Present	3.8413	3.6386	3.3173	3.8413	3.8283	3.8066

Eq. (30) represents the nonlinear dynamical response of FGM sandwich beams with pores, geometric imperfection and elastically restrained ends in thermal environments. The dynamic deflection–transient time curves are plotted by means of fourth-order Runge-Kutta method with initial conditions $W(0) = 0$ and $\dot{W}(0) = 0$. In case the geometrical nonlinearity and transverse load are neglected, the above Eq. (30) reduces to

$$I_1 h \frac{d^2 \bar{W}}{dt^2} + \left(a_{11} + \mu a_{21} + 2\mu^2 a_{31} - a_{41}\Delta T\right)\bar{W} = \mu a_{41}\Delta T \tag{32}$$

The above equation describes the linear free vibration of FGM sandwich beams with porosity, geometric imperfection and tangentially constrained ends in the thermal environments. The natural frequencies of FGM sandwich beams are determined by formula

$$\omega_L = \sqrt{\frac{a_{11} + \mu a_{21} + 2\mu^2 a_{31} - a_{41}\Delta T}{I_1 h}} \tag{33}$$

5. Numerical results and discussion

This section carries out parametric studies to examine different effects on the nonlinear transient response (NTR) of FGM sandwich beams made of Si_3N_4 and SUS304. The temperature dependent properties of Si_3N_4 and SUS304 such as elastic modulus and TEC have been exhibited in the preceding studies, e.g. [1, 15], and not displayed here for the sake of brevity. The mass densities of Si_3N_4 and SUS304 are $\rho_c = 2370$ kg/m³ and $\rho_m = 8166$ kg/m³, respectively. The effective Poisson coefficient of Si_3N_4/SUS304 is taken to be $\nu = 0.28$ [15].

5.1. Comparative study

The present subsection performs two comparisons to check the accuracy of the present formulations. Because there is no preceding study on the nonlinear dynamic response of FGM sandwich beams with porosity and elastically constrained ends, the comparisons are performed for single–layered FGM beams with special cases of in-plane ends condition. The results for single–layered FGM beams are attained by specializing the present formulations of sandwich beams of type B for the case of $h_f = 0$. First, the linear free vibration of FGM beams with porosity and immovable ends placed at room temperature is considered. This problem was also treated by Turan et al. [38] using analytical, finite element, and artificial neural network methods on the basis of the FSDBT and by Hadji et al. [50] employing the Navier–type solution within the framework of an improved version of higher order shear deformation beam theory (HSDBT). In the first comparison, the FGM beam is made of aluminum (Al) and alumina (Al_2O_3) with properties are $E_m = 70$ GPa, $\rho_m = 2702$ kg/m³, $\nu_m = 0.3$ for Al and $E_c = 380$ GPa, $\rho_c = 3960$ kg/m³, $\nu_c = 0.3$ for Al_2O_3 [38]. In comparative studies, the non-dimensional natural frequencies of FGM beams are evaluated as follows

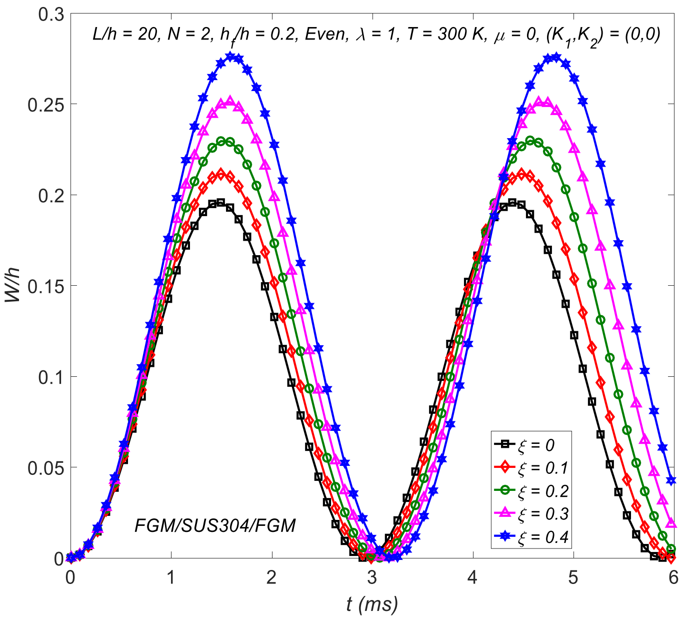
$$\bar{\omega} = \omega_L \frac{L^2}{h}\sqrt{\frac{\rho_m}{E_m}} \tag{34}$$

Non-dimensional fundamental natural

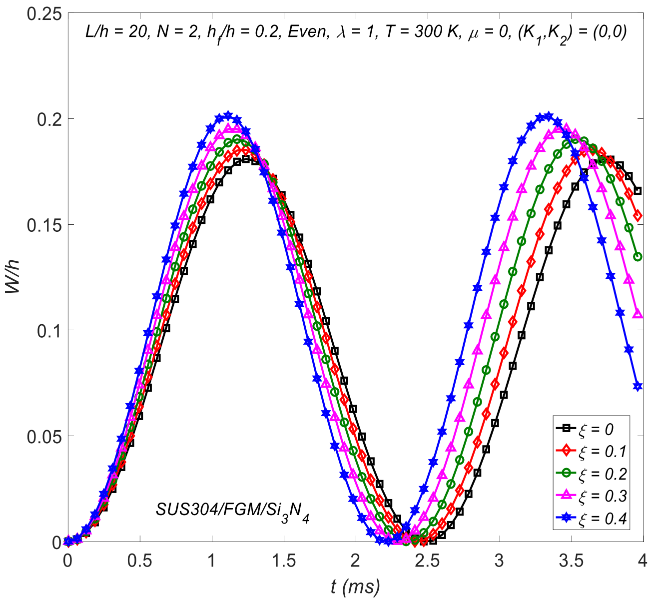
frequencies calculated by the present study are given in Table 1 in comparison with those of works [38, 50]. It can be recognized that a very good agreement is obtained in this comparative study.

Table 2. Comparison of dimensionless fundamental natural frequencies $\bar{\omega}$ of simply supported Si_3N_4/SUS304 beams with immovable ends in a thermal environment ($L/h=20$, $\Delta T=80$ K, $m=1$, $\lambda=1$).

Type	ξ	Source	N				
			0	0.5	1	2	5
Even	0	Ebrahimi and Jafari [35]	5.4349	3.4341	2.8884	2.4999	2.1873
		Present	5.4393	3.4373	2.8912	2.5027	2.1900
	0.1	Ebrahimi and Jafari [35]	6.1011	3.6245	3.0086	2.5829	2.2504
		Present	6.1060	3.6280	3.0117	2.5858	2.2532
	0.2	Ebrahimi and Jafari [35]	7.1036	3.8301	3.1215	2.6493	2.2933
		Present	7.1093	3.8339	3.1248	2.6523	2.2964
Uneven	0	Ebrahimi and Jafari [35]	5.4349	3.4341	2.8884	2.4999	2.1873
		Present	5.4393	3.4373	2.8912	2.5027	2.1900
	0.1	Ebrahimi and Jafari [35]	5.8231	3.5981	3.0175	2.6100	2.2871
		Present	5.8285	3.6019	3.0208	2.6132	2.2904
	0.2	Ebrahimi and Jafari [35]	6.2757	3.7669	3.1456	2.7163	2.3817
		Present	6.2823	3.7713	3.1495	2.7201	2.3856



(a)

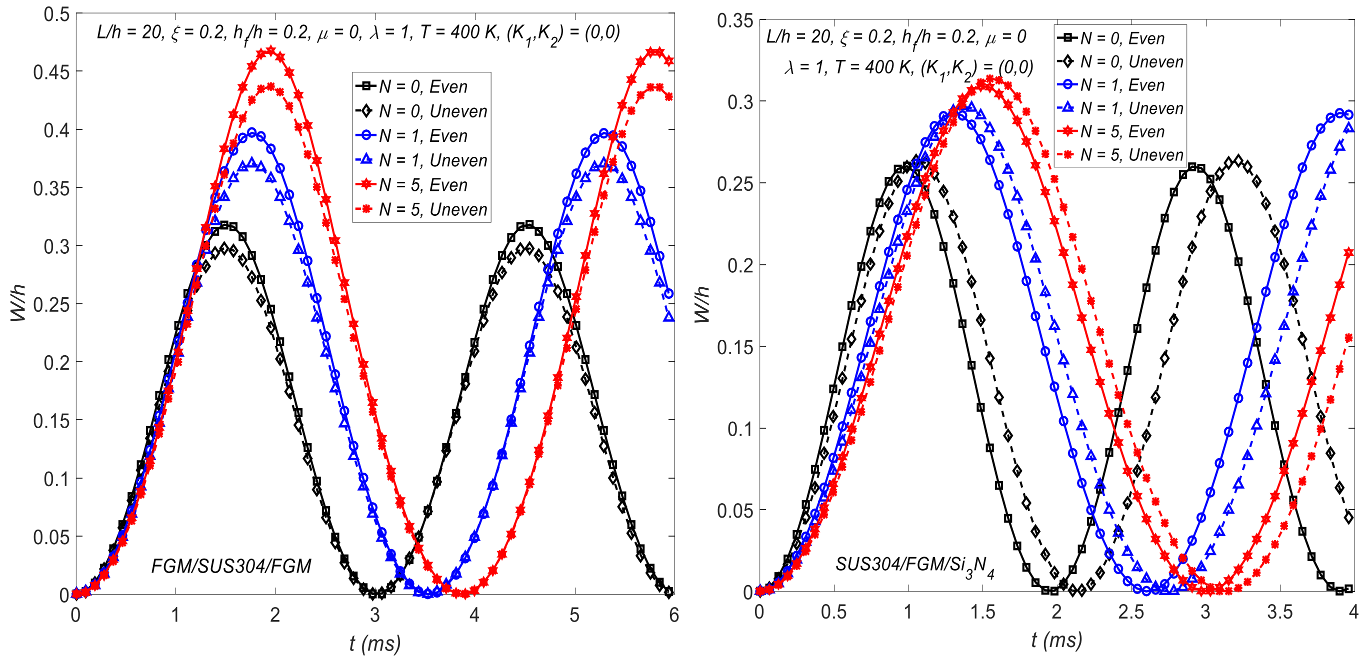


(b)

Fig. 4. Effects of ξ on the NTR of sandwich beams with immovable ends and even porosity: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

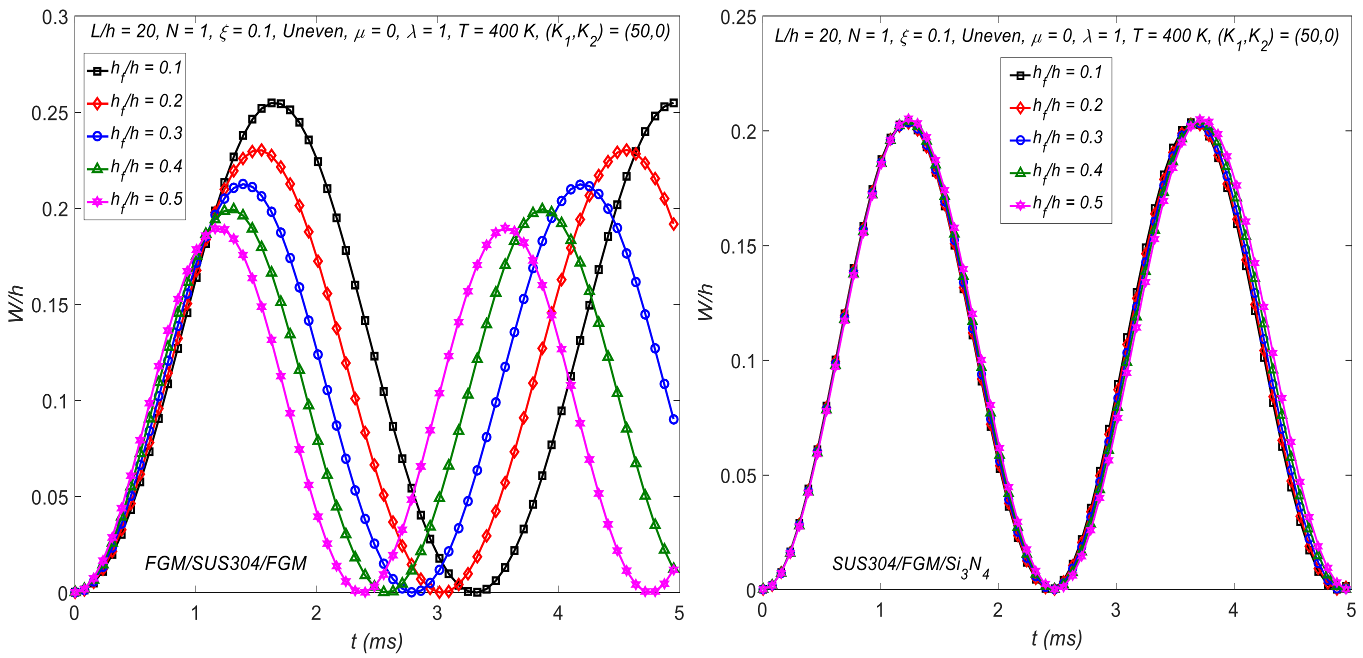
In the second comparison, the linear free vibration of a single–layered Si_3N_4/SUS304 beam with porosity and immovable ends in a thermal environment is considered. This problem was also dealt with by Ebrahimi and Jafari [35] employing the HSDBT and Navier–type solution. The non-dimensional values of fundamental natural frequencies $\bar{\omega}$ obtained by the present approach are shown in Table 2 along with those of Ebrahimi and Jafari [35]. Again, it is clear that a very good agreement is achieved in this comparison.

5.2. Nonlinear transient response



(a) (b)

Fig. 5. Effects of N index and porosity distribution pattern on the NTR of sandwich beams with immovable ends in a thermal environment: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

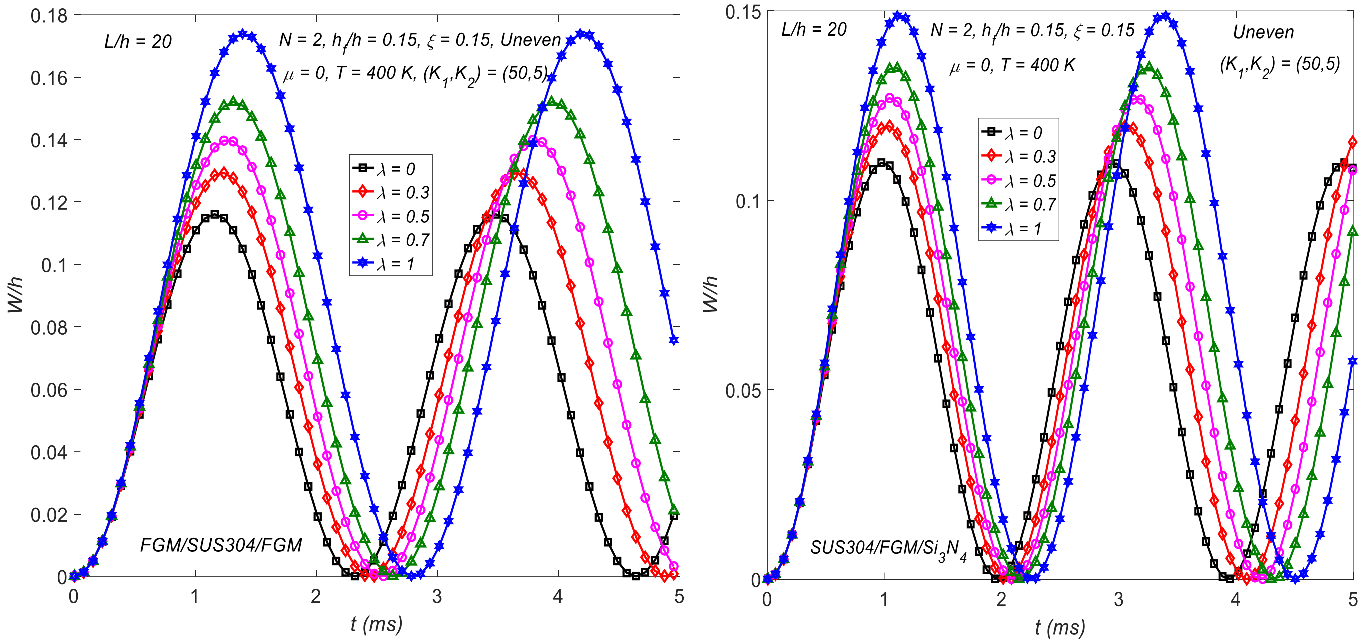


(a) (b)

Fig. 6. Impact of h_f / h ratio on nonlinear transient path of sandwich beams with immovable ends supported by a Winkler foundation in a thermal environment: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

In the followings, the nonlinear transient response (NTR) of FGM sandwich beams of type A (FGM/SUS304/FGM) and type B (SUS304/FGM/Si_3N_4) resting on elastic foundations will be analyzed in graphical form. The total thickness of sandwich beams is $h = 0.02\,\text{m}$, intensity of uniform transverse load is $q = 10^6\,\text{N/m}$, and mode shape is $m = 1$. The beams are assumed to be of perfect geometry ($\mu = 0$) and placed at reference temperature condition ($T = 300\,\text{K}$), unless otherwise stated.

First, the effects of volume fraction of pores ξ on the NTR of sandwich beams with immovable ends and even porosity are shown in Fig. 4. It is recognized that the porosity has deteriorative influences on the dynamic bending behavior of sandwich beams. Specifically, the amplitude of dynamic transverse displacement is pronouncedly increased due to increase in the value of ξ. It seems that the effects of the ξ on the NTR of the sandwich beams of type A are more significant. Besides, the vibration periods of sandwich beams of types A and B are increased and decreased, respectively, when the FGM layers are more porous.



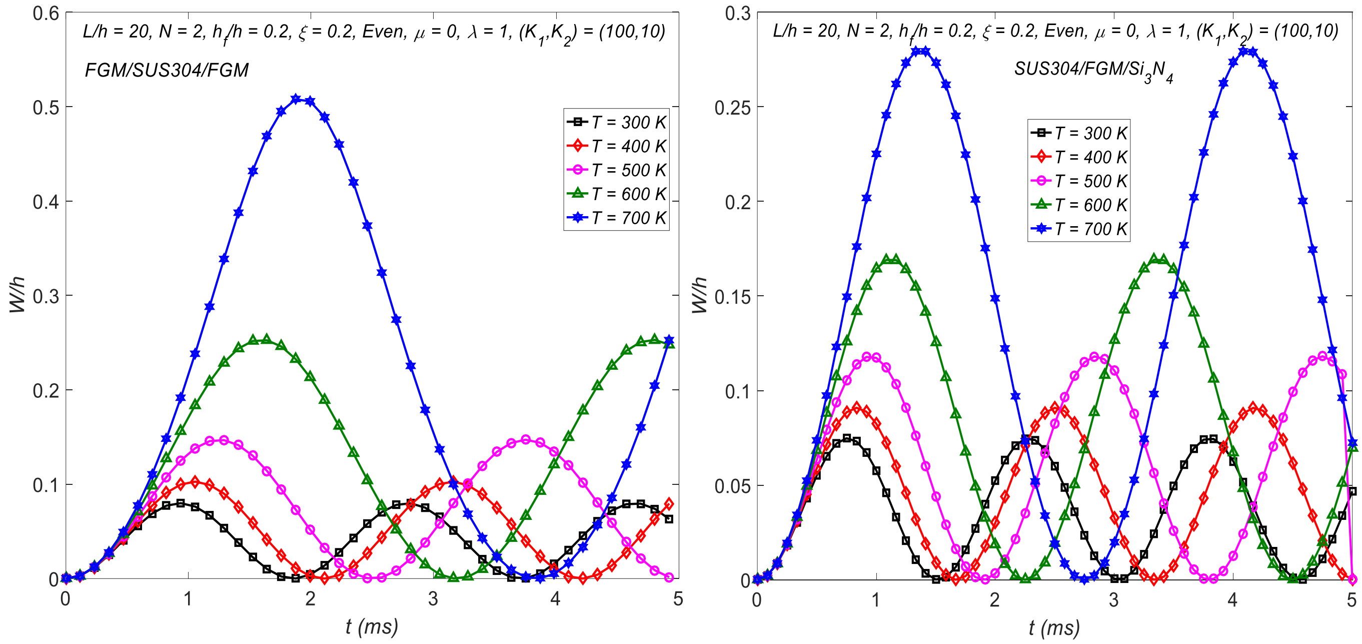
(a) (b)

Fig. 7. Influences of tangential ends constraints on the NTR of FGM sandwich beams with uneven porosity supported by a Winkler–Pasternak foundation in a thermal environment: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

Next, the effects of volume fraction index N and type of porosity distribution on the NTR of sandwich beams with immovable ends in a thermal environment ($T = 400\,\text{K}$) are graphed in Fig. 5. Evidently, the dynamic deflection of beams are augmented as the N index is larger. In another expression, the dynamic bending resistance capability of sandwich beams is substantially weaker when the ceramic percentage in the FGM layers is reduced. Moreover, it is realized that the vibration period of sandwich beams is raised as a result of the enhancement of the N index.

Subsequently, the influences of h_f / h ratio of thickness of face sheets to total thickness on the NTR of sandwich beams with immovable ends and uneven porosity resting on a Winkler foundation are illustrated in Fig. 6.This figure demonstrates that the h_f / h ratio has significant influence on the NTR of beam of type A and both dynamic deflection and vibration period of FGM/SUS304/FGM beams are considerably decreased when the h_f / h ratio is increased. This result reflects that the dynamic bending resistance capacity of FGM/SUS304/FGM beams is pronouncedly stronger when the FGM skin layers are thicker. Meanwhile, the h_f/h ratio has a very slight impact on the NTR of sandwich beams of type B. It is seen that the intensity of dynamic deflection and period of vibration of SUS304/FGM/Si_3N_4 beams are marginally increased when the thickness of homogeneous face layers is larger.

Next parametric study is illustrated in Fig. 7 illustrating the effects of axial constraints of ends (i.e. λ parameter) on the NTR of sandwich beams with uneven porosity reposing on a Winkler–Pasternak foundation $(K_1 = 50, K_2 = 5)$ in a thermal environment $(T = 400\,K)$. As can be observed, the amplitude of dynamic displacement and vibration period are substantially increased when the ends of beam are restrained more strictly, i.e. λ parameter is larger. It is attributed to thermal forces induced at restrained ends making the bending resistance ability of sandwich beams weaker.

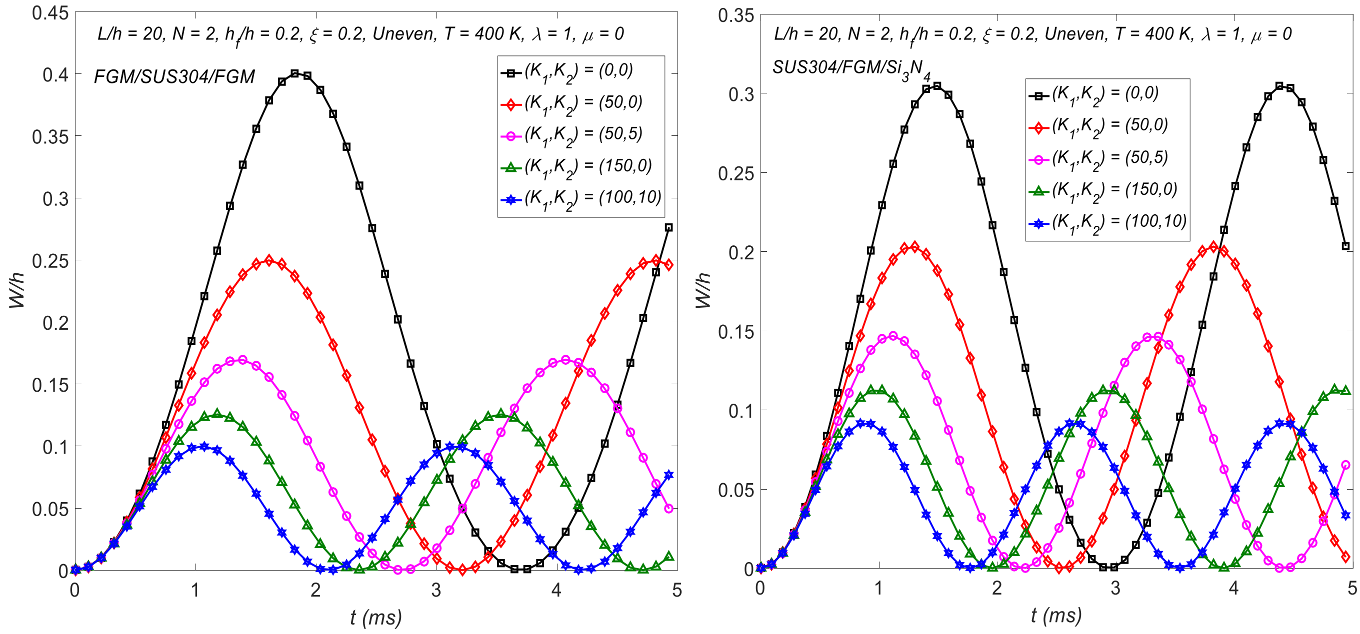


(a) (b)

Fig. 8. Effects of thermal environments on the NTR of sandwich beams with immovable ends and even porosity resting on a Winkler–Pasternak foundation: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

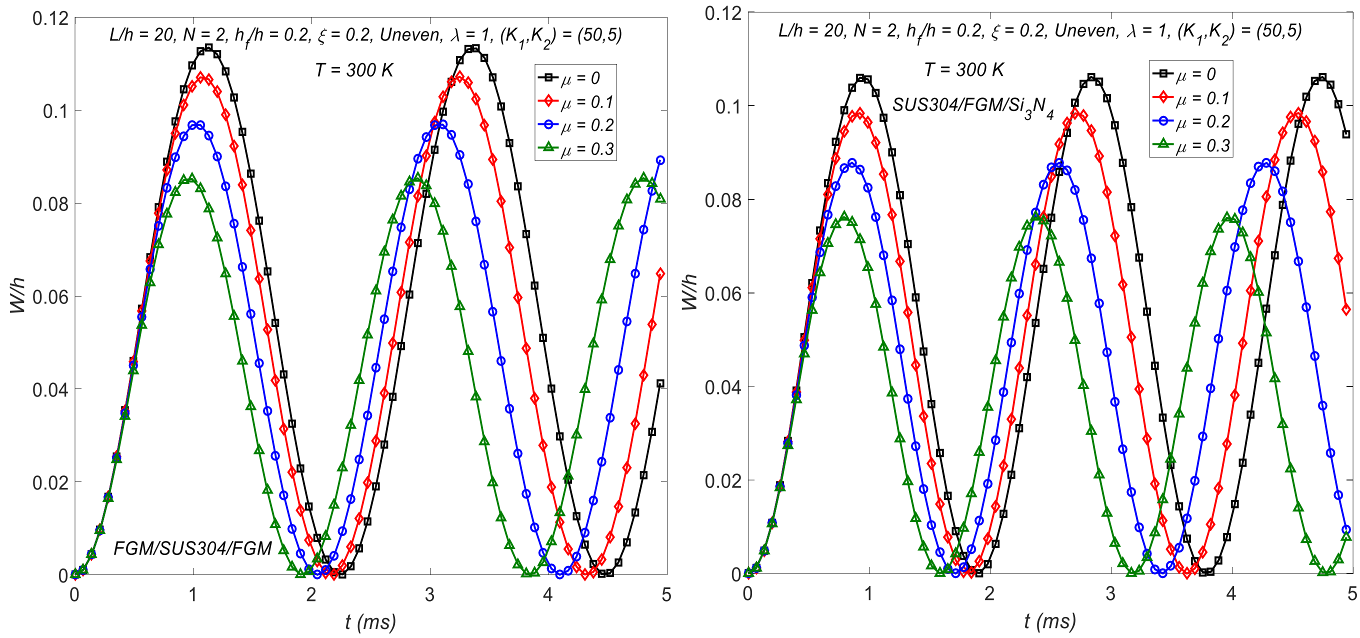
The influences of high temperatures T on the NTR of sandwich beams with immovable ends and even porosity resting on a Winkler–Pasternak elastic foundation $(K_1 = 100, K_2 = 10)$ are analyzed in Fig. 8. It is evident that elevated temperature renders the dynamic transverse displacement deeper and period of forced vibration larger. When the ends are tangentially restrained, the thermally induced forces at ends are more intense if the temperature is more elevated. In this way, the elevated temperature detrimentally affects the dynamic bending resistance capacity of sandwich beams with restrained ends.

In the Fig. 9, the influences of elastic foundations (through the dimensionless stiffness parameters K_1, K_2) on the NTR of sandwich beams with immovable ends $(\lambda = 1)$ and uneven porosity in a thermal environment are analyzed. Obviously, the support of elastic foundation brings to evident effectiveness in improving the dynamic bending resistance capability of sandwich beams. More concretely, the amplitude of dynamic transverse displacement is substantially reduced when the stiffness parameters of foundations become larger.



(a) (b)

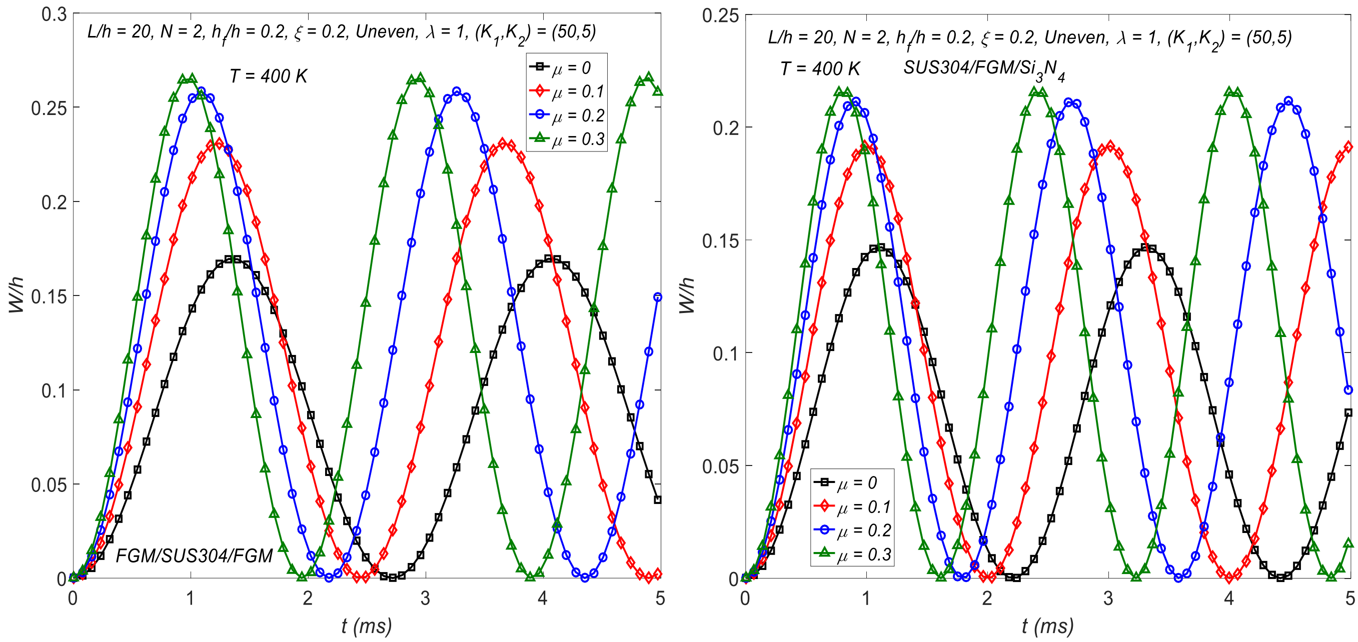
Fig. 9. Effects of foundation stiffness on the NTR of sandwich beams with immovable ends and uneven porosity in a thermal environment: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.



(a) (b)

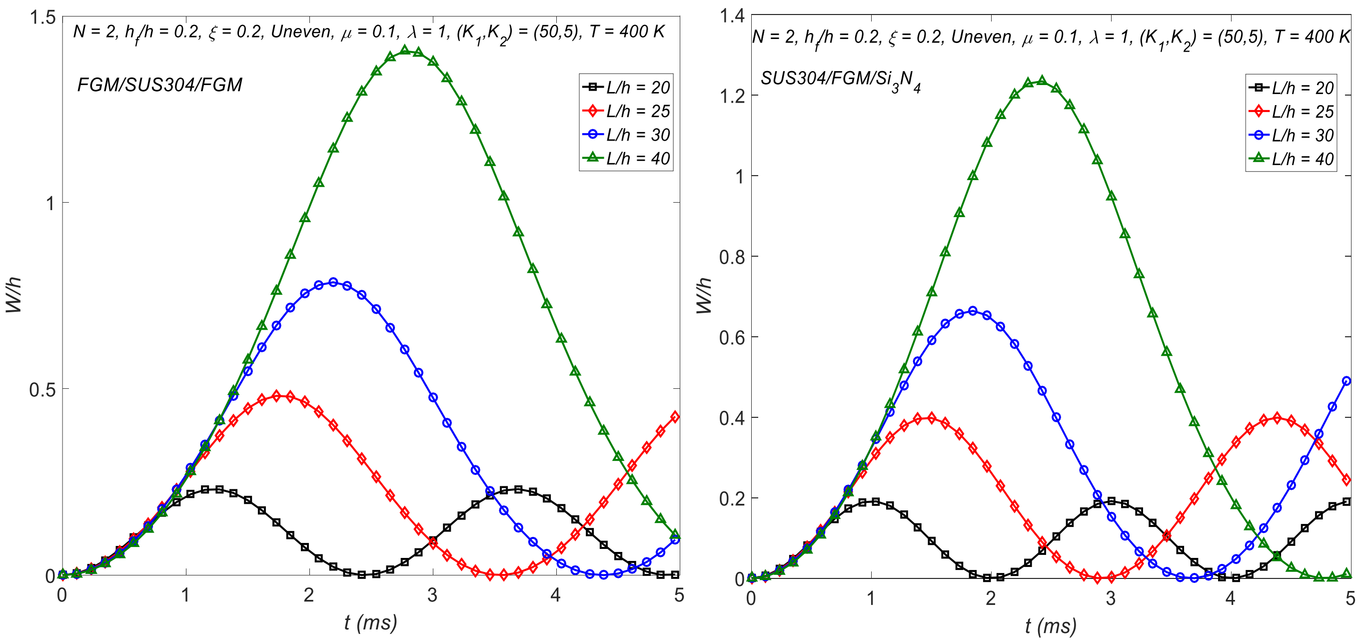
Fig. 10. Effects of geometric imperfection on the NTR of sandwich beams with immovable ends and uneven porosity at room temperature: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

Next, the effects of geometric imperfection on the NTR of sandwich beams with immovable ends and uneven porosity placed at room temperature are plotted in Fig. 10. It is realized that the geometric imperfection has a beneficial effect on the bending resistance of sandwich beams at room temperature and intensity of dynamic displacement is remarkably lowered in case the size μ of geometric imperfection becomes larger. It is found that additional deflection of sandwich beams only subjected to mechanical load is reduced due to the presence of geometric imperfection.



(a) (b)

Fig. 11. Effects of geometric imperfection on the NTR of sandwich beams with immovable ends and uneven porosity at elevated temperature: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.



(a) (b)

Fig. 12. Effects of slenderness ratio on the NTR of sandwich beams with immovable ends and uneven porosity in a thermal environment: a) FGM/SUS304/FGM and b) SUS304/FGM/Si_3N_4.

In contrast, the maximum value of dynamic displacement and period of forced vibration of sandwich beams placed at a high temperature ($T = 400\,\mathrm{K}$) are increased and dropped, respectively, as the size of geometric imperfection is larger, as demonstrated in Fig. 11. Once again, the presence of thermal forces at restrained ends makes the effect of initial imperfection more negative.

Finally, the influences of slenderness (i.e.

L/h ratio) on the NTR of sandwich beams with immovable ends and uneven porosity resting on a Winkler–Pasternak foundation in a thermal environment are graphed in Fig. 12. It can be found that the dynamic transverse displacement and period of forced vibration are significantly augmented as the L/h ratio becomes larger. Thus, the dynamic bending resistance capacity of sandwich beams become evidently lower when the beams are slenderer.

6. Conclusions

A semi–analytical study on the simultaneous effects of pores, initial geometric imperfection, elastic foundations, elastic restraints of ends and elevated temperatures on the NTR of FGM sandwich beams has been presented. From the above analyses, the following conclusions are attained:

i) The presence of pores has negative effect on the dynamic bending resistance capacity of sandwich beams and the amplitude of deflection is pronouncedly increased as the volume fraction of pores in FGM layers is larger.

ii) The axial constraints of ends make the dynamic deflection of porous FGM sandwich beams exposed to elevated temperatures deeper. Besides, the dynamical deflection of sandwich beams with restrained ends is increased as temperature is more elevated.

iii) Initial geometric imperfections have beneficial and deteriorative influences on the dynamic bending resistance capability of porous FGM sandwich beams with restrained ends exposed to room and elevated temperatures, respectively.

iv) The dynamic bending resistance capacity of porous FGM sandwich beams is significantly strengthened owing to the presence of elastic foundations. The amplitude of dynamic deflection is prominently reduced when the stiffness parameters of foundations are larger.

Declaration of Generative AI

During the preparation of this work, the authors do not use Generative AI and AI-assisted technologies tools in order to generate the content of the manuscript.

References

[1] N. Wattanasakulpong, B.G. Prusty, D.W. Kelly. (2011). Thermal buckling and elastic vibration of third-order shear deformable functionally graded beams. *International Journal of Mechanical Sciences*, 53, 734-743. https://doi.org/10.1016/j.ijmecsci.2011.06.005

[2] K.K. Pradhan, S. Chakraverty. (2013). Free vibration of Euler and Timoshenko functionally graded beams by Rayleigh–Ritz method. *Composites Part B: Engineering*, 51, 175-184. https://doi.org/10.1016/j.compositesb.2013.02.027

[3] M. Şimşek. (2010). Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories. *Nuclear Engineering and Design*, 240(4), 697-705. https://doi.org/10.1016/j.nucengdes.2009.12.013

[4] M. Azadi. (2011). Free and forced vibration analysis of FG beam considering temperature dependency of material properties. *Journal of Mechanical Science and Technology*, 25(1), 69-80. https://doi.org/10.1007/s12206-010-1015-y

[5] G.H. Rahimi, M.S. Gazor, M. Hemmatnezhad, H. Toorani. (2013). On the postbuckling and free vibrations on FG Timoshenko beams. *Composite Structures*, 95, 247-253. https://doi.org/10.1016/j.compstruct.2012.07.034

[6] H.T. Thai, T.P. Vo. (2012). Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories. *International Journal of Mechanical Sciences*, 62(1), 57-66. https://doi.org/10.1016/j.ijmecsci.2012.05.014

[7] X. Wang, S. Li. (2016). Free vibration analysis of functionally graded material beams based on Levinson beam theory. *Applied Mathematics and Mechanics*, 37(7), 861-878. https://doi.org/10.1007/s10483-016-2094-9

[8] A.E. Alshorbagy, M.A. Eltaher, F.F. Mahmoud.

(2011). Free vibration characteristics of a functionally graded beam by finite element method. *Applied Mathematical Modelling*, 35(1), 412-425. https://doi.org/10.1016/j.apm.2010.07.006

[9] A. Karamanli. (2018). Free vibration analysis of two directional functionally graded beams using a third order shear deformation theory. *Composite Structures*, 189, 127-136. https://doi.org/10.1016/j.compstruct.2018.01.060

[10] F. Pang, C. Gao, H. Li, D. Jia, X. Wang, X. Miao. (2022). Vibration analysis of FG beams under arbitrary load with general boundary conditions: Theoretical and experimental comparative research. *Thin-Walled Structures*, 179, 109605. https://doi.org/10.1016/j.tws.2022.109605

[11] A. Fallah, M.M. Aghdam. (2011). Nonlinear free vibration and post-buckling analysis of functionally graded beams on nonlinear elastic foundation. *European Journal of Mechanics A/Solids*, 30(4), 571-583. https://doi.org/10.1016/j.euromechsol.2011.01.005

[12] Y. Fu, J. Wang, Y. Mao. (2012). Nonlinear analysis of buckling, free vibration and dynamic stability for the piezoelectric functionally graded beams in thermal environment. *Applied Mathematical Modelling*, 36(9), 4324-4340. https://doi.org/10.1016/j.apm.2011.11.059

[13] H. Yaghoobi, M. Torabi. (2013). Post-buckling and nonlinear free vibration analysis of geometrically imperfect functionally graded beams resting on nonlinear elastic foundation. *Applied Mathematical Modelling*, 37(18-19), 8324-8340. https://doi.org/10.1016/j.apm.2013.03.037

[14] D.G. Zhang. (2014). Thermal post-buckling and nonlinear vibration analysis of FGM beams based on physical neutral surface and high order shear deformation theory. *Meccanica*, 49, 283-293. https://doi.org/10.1007/s11012-013-9793-9

[15] H.S. Shen, Z.X. Wang. (2014). Nonlinear analysis of shear deformable FGM beams resting on elastic foundations in thermal environments. *International Journal of Mechanical Sciences*, 81, 195-206. https://doi.org/10.1016/j.ijmecsci.2014.02.020

[16] A. Shooshtari, M. Rafiee. (2011). Nonlinear forced vibration analysis of clamped functionally graded beams. *Acta Mechanica*, 221, 23-38. https://doi.org/10.1007/s00707-011-0491-1

[17] M. Hemmatnezhad, R. Ansari, G.H. Rahimi. (2013). Large-amplitude free vibrations of functionally graded beams by means of a finite element formulation. *Applied Mathematical Modelling*, 37(18-19), 8495-8504. https://doi.org/10.1016/j.apm.2013.03.055

[18] A.S. Kanani, H. Niknam, A.R. Ohadi, M.M. Aghdam. (2014). Effect of nonlinear elastic foundation on large amplitude free and forced vibration of functionally graded beams. *Composite Structures*, 115, 60-68. https://doi.org/10.1016/j.compstruct.2014.04.003

[19] K. Xie, Y. Wang, X. Fan, T. Fu. (2020). Nonlinear free vibration analysis of functionally graded beams by using different shear deformation theories. *Applied Mathematical Modelling*, 77, 1860-1880. https://doi.org/10.1016/j.apm.2019.09.024

[20] K. Xie, Y. Wang, T. Fu. (2020). Nonlinear vibration analysis of third-order shear deformable functionally graded beams by a new method based on direct numerical integration technique. *International Journal of Mechanics and Materials in Design*, 16, 839-855. https://doi.org/10.1007/s10999-020-09493-y

[21] A.M. Zenkour. (2005). A comprehensive analysis of functionally graded sandwich plates: Part 2 – Buckling and free vibration. *International Journal of Solids and Structures*, 42(18-19), 5243-5258. https://doi.org/10.1016/j.ijsolstr.2005.02.016

[22] A.M.A. Neves, A.J.M. Ferreira, E. Carrera, M. Cinefra, C.M.C. Roque, R.M.N. Jorge, C.M.M. Soares. (2013). Static, free vibration and buckling analysis of isotropic and sandwich functionally graded plates using a quasi-3D higher-order shear deformation theory and meshless technique. *Composites Part B: Engineering*, 44(1), 657-674. https://doi.org/10.1016/j.compositesb.2012.01.089

[23] N.E. Meiche, A. Tounsi, N. Ziane, I. Mechab, E.A.A. Bedia. (2011). A new hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate. *International Journal of Mechanical Sciences*, 53(4), 237-247. https://doi.org/10.1016/j.ijmecsci.2011.01.004

[24] A. Bessaim, M.S. Houari, A. Tounsi, S. Mahmoud, E.A.A. Bedia. (2013). A new higher order shear and normal deformation theory for the static and free vibration analysis of sandwich plates with functionally graded isotropic face sheets. *Journal of Sandwich Structures and Materials*, 15(6), 671-703. https://doi.org/10.1177/1099636213498888

[25] H.S. Shen, S.R. Li. (2008). Postbuckling of sandwich plates with FGM face sheets and temperature-dependent properties. *Composites Part B: Engineering*, 39(2), 332-344. https://doi.org/10.1016/j.compositesb.2007.01.004

[26] Z.-X. Wang, H.-S. Shen. (2013). Nonlinear dynamic response of sandwich plates with FGM face sheets resting on elastic foundations in thermal environments. *Ocean Engineering*, 57, 99-110. https://doi.org/10.1016/j.oceaneng.2012.09.004

[27] H.V. Tung. (2015). Thermal and thermomechanical postbuckling of FGM sandwich plates resting on elastic foundations with tangential edge constraints and temperature dependent properties. *Composite Structures*, 131, 1028-1039. https://doi.org/10.1016/j.compstruct.2015.06.043

[28] H.V. Tung. (2018). Nonlinear thermomechanical response of pressure-loaded doubly curved functionally graded material sandwich panels in thermal environments including tangential edge constraints. *Journal of Sandwich Structures and Materials*, 20(8), 974-1008. https://doi.org/10.1177/1099636216684312

[29] L. Iurlaro, M. Gherlone, M. Di Sciuva. (2014). Bending and free vibration analysis of functionally graded sandwich plates using the refined zigzag theory. *Journal of Sandwich Structures and Materials*, 16(6), 669-699. https://doi.org/10.1177/1099636214548618

[30] Z. Su, G. Jin, Y. Wang, X. Ye. (2016). A general Fourier formulation for vibration analysis of functionally graded sandwich beams with arbitrary boundary condition and resting on elastic foundations. *Acta Mechanica*, 227, 1493-1514. https://doi.org/10.1007/s00707-016-1575-8

[31] N. Wattanasakulpong, V. Ungbhakorn. (2014). Linear and nonlinear vibration analysis of elastically restrained ends FGM beams with porosities. *Aerospace Science and Technology*, 32(1), 111-120. https://doi.org/10.1016/j.ast.2013.12.002

[32] N. Wattanasakulpong, A. Chaikittiratana. (2015). Flexural vibration of imperfect functionally graded beams based on Timoshenko beam theory: Chebyshev collocation method. *Meccanica*, 50, 1331-1342. https://doi.org/10.1007/s11012-014-0094-8

[33] N.V. Thinh, H.V. Tung. (2024). Thermal buckling and postbuckling of porous FGM beams with elastically restrained ends resting on nonlinear elastic foundations. *International Journal of Computational Materials Science and Engineering*, 2450033. https://doi.org/10.1142/S2047684124500337

[34] N.D. Anh, N.V. Thinh, H.V. Tung. (2024).

Nonlinear vibration analysis of geometrically imperfect FGM Timoshenko beams with porosity and elastically restrained ends in thermal environments. *Mechanics Based Design of Structures and Machines*, 52(12), 10148-10169. https://doi.org/10.1080/15397734.2024.2353876

[35] F. Ebrahimi, A. Jafari. (2016). A higher–order thermomechanical vibration analysis of temperature–dependent FGM beams with porosities. *Journal of Engineering,* 2016, 9561504. https://doi.org/10.1155/2016/9561504

[36] S. Zghal, D. Ataoui, F. Dammak. (2022). Static bending analysis of beams made of functionally graded porous materials. *Mechanics Based Design of Structures and Machines*, 50(3), 1012-1029. https://doi.org/10.1080/15397734.2020.1748053

[37] S.D. Akbas. (2017). Thermal effects on the vibration of functionally graded deep beams with porosity. *International Journal of Applied Mechanics*, 9(5), 1750076. https://doi.org/10.1142/S1758825117500764

[38] M. Turan, E.U. Yaylaci, M. Yaylaci. (2023). Free vibration and buckling of functionally graded porous beams using analytical, finite element, and artificial neural network methods. *Archive of Applied Mechanics*, 93, 1351-1372. https://doi.org/10.1007/s00419-022-02332-w

[39] V.T. Long, H.V. Tung. (2026). Nonlinear free vibration of FGM sandwich plates with porosity under uniaxial compressive loads in thermal environments. *Noise & Vibration Worldwide*, 57(6-7), 468-485. https://doi.org/10.1177/09574565261419561

[40] H.V. Tung, V.T. Long, D.T.N. Thu. (2026). Thermally induced nonlinear instability of functionally graded plates including pores and elastic edge restraint. *Journal of Science and Transport Technology*, 6(2), 37-55. https://doi.org/10.58845/jstt.utt.2026.en.6.2.37-55

[41] N.V. Thinh, B.N. Son, H.V. Tung. (2026). Nonlinear dynamical response of FG sandwich plates with pores and flexibly constrained boundaries in thermal environments. *Journal of Science and Transport Technology*, 6(2), 1-18. https://doi.org/10.58845/jstt.utt.2026.en.6.2.1-18

[42] H.V. Tung, N.V. Thinh. (2025). Nonlinear thermomechanical transient response of initially stressed FGM shallow shell panels including porosity and geometric imperfection. *Journal of Aerospace Engineering*, 38(5), 04025061. https://doi.org/10.1061/JAEEEZ.ASENG-6120

[43] N.V. Thinh, H.V. Tung. (2026). Nonlinear thermomechanical transient response of FGM circular plates and shallow spherical caps with porosities and tangentially restrained edges. *Journal of Aerospace Engineering*, 39(2), 04025140. https://doi.org/10.1061/JAEEEZ.ASENG-6480

[44] J.N. Reddy. (2004). Mechanics of laminated composite plates and shells: theory and analysis. 2nd Edition. *CRC Press*. https://doi.org/10.1201/b12409

[45] L. Librescu, W. Lin. (1997). Vibration of thermomechanically loaded flat and curved panels taking into account geometric imperfections and tangential edge restraints. *International Journal of Solids and Structures*, 34(17), 2161-2181. https://doi.org/10.1016/S0020-7683(96)00025-X

[46] N.V. Thinh, H.V. Tung. (2024). Thermo-electrical postbuckling behavior of carbon nanotubes-reinforced composite beams with piezoelectric layers and tangentially restrained ends. *Journal of Thermoplastic Composite Materials*, 37(8), 2657-2679. https://doi.org/10.1177/08927057231216739

[47] N.V. Thinh, H.V. Tung. (2024). Nonlinear dynamical characteristics of carbon nanotube–reinforced composite beams with piezoelectric

actuators and elastically restrained ends under thermo–electro–mechanical loads. *Journal of Thermoplastic Composite Materials*, 37(11), 3460-3491. https://doi.org/10.1177/08927057241233187

[48] N.D. Anh, N.V. Thinh, H.V. Tung. (2023). Thermoelastic nonlinear vibration and dynamical response of geometrically imperfect carbon nanotube-reinforced composite plates on elastic foundations including tangential edge constraints. *Journal of Thermoplastic Composite Materials*, 37, 1067-1093. DOI: 10.1177/08927057231191454

[49] J.G. Eisley. (1964). Nonlinear vibration of beams and rectangular plates. *Journal of Applied Mathematics and Physics (ZAMP)*, 15, 167-175. https://doi.org/10.1007/BF01602658

[50] L. Hadji, N. Zouatnia, F. Bernard. (2019). An analytical solution for bending and free vibration responses of functionally graded beams with porosities: Effect of the micromechanical models. *Structural Engineering and Mechanics*, 69(2), 231-241. https://doi.org/10.12989/sem.2019.69.2.231